





# Cohomology for quantum groups via the geometry of the nullcone

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## Abstract

Let  $\zeta$  be a complex  $\ell$ th root of unity for an odd integer  $\ell > 1$ . For any complex simple Lie algebra  $\mathfrak{g}$ , let  $u_\zeta = u_\zeta(\mathfrak{g})$  be the associated “small” quantum enveloping algebra. This algebra is a finite dimensional Hopf algebra which can be realized as a subalgebra of the Lusztig (divided power) quantum enveloping algebra  $U_\zeta$  and as a quotient algebra of the De Concini–Kac quantum enveloping algebra  $\mathcal{U}_\zeta$ . It plays an important role in the representation theories of both  $U_\zeta$  and  $\mathcal{U}_\zeta$  in a way analogous to that played by the restricted enveloping algebra  $u$  of a reductive group  $G$  in positive characteristic  $p$  with respect to its distribution and enveloping algebras. In general, little is known about the representation theory of quantum groups (resp., algebraic groups) when  $l$  (resp.,  $p$ ) is smaller than the Coxeter number  $h$  of the underlying root system. For example, Lusztig’s conjecture concerning the characters of the rational irreducible  $G$ -modules stipulates that  $p \geq h$ . The main result in this paper provides a surprisingly uniform answer for the cohomology algebra  $H^\bullet(u_\zeta, \mathbb{C})$  of the small quantum group. When  $\ell > h$ , this cohomology algebra has been calculated by Ginzburg and Kumar [GK]. Our result requires powerful tools from complex geometry and a detailed knowledge of the geometry of the nullcone of  $\mathfrak{g}$ . In this way, the methods point out difficulties present in obtaining similar results for the restricted enveloping algebra  $u$  in small characteristics, though they do provide some clarification of known results there also. Finally, we establish that if  $M$  is a finite dimensional  $u_\zeta$ -module, then  $H^\bullet(u_\zeta, M)$  is a finitely generated  $H^\bullet(u_\zeta, \mathbb{C})$ -module, and we obtain new results on the theory of support varieties for  $u_\zeta$ .

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## Introduction

Let  $\mathfrak{g}_F$  be a finite dimensional, restricted Lie algebra (as defined by Jacobson) over an algebraically closed field  $F$  of positive characteristic  $p$ , with restriction map  $x \mapsto x^{[p]}$ ,  $x \in \mathfrak{g}_F$ . The restricted enveloping algebra  $u := u(\mathfrak{g}_F)$  of  $\mathfrak{g}_F$  is a finite dimensional cocommutative Hopf algebra. In general, the cohomology algebra  $H^\bullet(u, F)$  is difficult to compute. However, Suslin-Friedlander-Bendel [SFB1, SFB2] proved that, putting  $A := H^{2\bullet}(u, F)$  (the commutative subalgebra of the cohomology algebra concentrated in even degrees), the (algebraic) scheme  $\text{Spec } A$  is homeomorphic to the closed subvariety  $\mathcal{N}_1(\mathfrak{g}_F) := \{x \in \mathfrak{g}_F : x^{[p]} = 0\}$ . We call  $\mathcal{N}_1(\mathfrak{g}_F)$  the restricted nullcone of  $\mathfrak{g}_F$ ; it is a closed subvariety of the full nullcone  $\mathcal{N}(\mathfrak{g}_F)$  which consists of all  $[p]$ -nilpotent elements in  $\mathfrak{g}_F$ .

When  $\mathfrak{g}_F$  is the Lie algebra of a reductive algebraic group  $G$  over  $F$ , the above results can be considerably sharpened. For example, if  $p > h$  (the Coxeter number of  $G$ ), then  $H^{2\bullet}(u, F) \cong F[\mathcal{N}_1(\mathfrak{g}_F)]$ , the coordinate algebra of  $\mathcal{N}_1(\mathfrak{g}_F)$  (cf. Friedlander-Parshall [FP2] and Andersen-Jantzen [AJ]). In addition, the condition  $p \geq h$  implies that  $\mathcal{N}_1(\mathfrak{g}_F) = \mathcal{N}(\mathfrak{g}_F)$ . However, when  $p \leq h$ , there is no known calculation of  $H^\bullet(u, F)$  (apart from some small rank cases). For all primes,  $\mathcal{N}_1(\mathfrak{g}_F)$  is an irreducible variety [NPV, (6.3.1) Cor.], [UGA2, Thm. 4.2] and is the closure of a  $G$ -orbit. These orbits have been determined in [CLNP], [UGA2, Thm. 4.2].

Now let  $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$  be a complex simple Lie algebra, and let  $U_\zeta = U_\zeta(\mathfrak{g})$  be the quantum enveloping algebra (Lusztig form) associated to  $\mathfrak{g}$  at a primitive  $l$ th root of unity  $\zeta \in \mathbb{C}$ . We regard  $U_\zeta$  as an algebra over  $\mathbb{C}$  obtained by base change from the quantum enveloping algebra over the cyclotomic field  $\mathbb{Q}(\zeta)$ . Here  $l > 1$  is an odd integer, not divisible by 3 when  $\mathfrak{g}$  has type  $G_2$ . The representation theory of  $U_\zeta$ , when  $l = p$ , models or approximates the representation theory of  $G$ . For example, part of the “Lusztig program” to determine the characters of the irreducible  $G$ -modules amounts to showing, when  $l = p \geq h$ , that the characters of the irreducible  $G$ -modules having high weights in the so-called Jantzen region coincide with the characters of the analogous irreducible modules for  $U_\zeta$ . This result has been proved by Andersen-Jantzen-Soergel [AJS] for all large  $p$  (no lower bound known except for small rank). Recently, Fiebig [Fie], improving upon the methods of [AJS], has provided a specific (large) bound on  $p$  for each root system sufficient for the validity of the Lusztig character formula. The original conjecture for  $p \geq h$  remains open.

For all  $l$ , Lusztig [L1] has also introduced an analog, denoted  $u_\zeta := u_\zeta(\mathfrak{g})$ , of the restricted enveloping algebra  $u$ . Like  $u$ ,  $u_\zeta$  is a finite dimensional Hopf algebra, though it is in general not cocommutative. It plays a role in the representation theory of  $U_\zeta$  much like that played by  $u$  in the representation theory of  $G$ . For  $l > h$ , Ginzburg-Kumar [GK] have calculated the cohomology algebra  $H^\bullet(u_\zeta, \mathbb{C})$ . By an exact analogy with the cohomology algebra of  $u$ , they prove that  $H^{2\bullet}(u_\zeta, \mathbb{C}) \cong \mathbb{C}[\mathcal{N}(\mathfrak{g})]$ , the coordinate algebra of the nullcone  $\mathcal{N}(\mathfrak{g})$  consisting of nilpotent elements in  $\mathfrak{g}$ . Recently, Arkhipov-Bezrukavnikov-Ginzburg [ABG, §1.4], taking [GK] as a method to pass from the representation theory of quantum groups to the geometry of the nullcone, provide a proof of Lusztig’s character formula for  $U_\zeta$  when  $l > h$ . These important connections have made the small quantum group  $u_\zeta$  an object of significant interest (cf. [ABGM], [AG], [Be], [Lac1, Lac2]).

This paper presents new results on the cohomology of  $u_\zeta$ . In particular, we compute the cohomology algebra  $H^\bullet(u_\zeta, \mathbb{C})$  in most of the remaining cases when  $l \leq h$ . Our results are explicitly described



in Section 1.2 below. We prove in Chapters 2–5 that  $H^{2\bullet+1}(u_\zeta, \mathbb{C}) = 0$ , while  $H^{2\bullet}(u_\zeta, \mathbb{C})$  is isomorphic to, in most cases, the coordinate algebra of an explicitly described closed subvariety  $\mathcal{N}(\Phi_0)$  of  $\mathcal{N}(\mathfrak{g})$  (constructed in a similar way to the variety  $\mathcal{N}_1(\mathfrak{g}_F)$  discussed above). In Chapter 2, we rigorously develop and present new results on the cohomology theory of parabolic subalgebras for quantum groups. The application of powerful tools from complex geometry represents at least one advantage that the quantum enveloping algebra situation (in characteristic zero) has over that in positive characteristic. In Chapters 3 and 5, we demonstrate how the Grauert-Riemenschneider theorem and the normality of certain orbit closures in  $\mathcal{N}(\mathfrak{g})$ , namely, the varieties  $\mathcal{N}(\Phi_0)$ , play a vital role in carrying out our cohomology calculations. Some of the major input in our calculations occurs in Chapter 4, where we analyze the combinatorics involving the multiplicity of the Steinberg module in certain cohomology modules related to unipotent radicals of parabolic subalgebras.

In Chapter 6, we prove that  $R := H^\bullet(u_\zeta, \mathbb{C})$  is a finitely generated  $\mathbb{C}$ -algebra. Also, we show that if  $M$  is a finite dimensional  $u_\zeta$ -module, then  $H^\bullet(u_\zeta, M)$  is finitely generated over  $R$ . Chapter 7 adapts some of our methods to the cohomology algebra  $H^\bullet(u, F)$  in positive characteristic, making some ad hoc computations in [AJ] more transparent. In particular, we can identify key vanishing results, known over  $\mathbb{C}$ , and as yet unproved in positive characteristic, which would be sufficient to extend the cohomology calculations to positive characteristic  $p$ .

Finally, by building on results in Chapter 6, we define support varieties in the quantum setting in Chapter 8 and exhibit some new calculations on support varieties. The theory of support varieties for the restricted Lie algebra  $\mathfrak{g}_F$  attached to a reductive group  $G$  in positive characteristic provides evidence for beautiful connections between the representation theory of  $G$  and the structure and geometry of the restricted nullcone  $\mathcal{N}_1(\mathfrak{g}_F)$  ([NPV], [CLNP], [UGA1], [UGA2]). The results of this paper strongly reinforce this expectation (as do those in [ABG] mentioned above).

A number of applications and further developments which heavily depend on the foundational results of this paper have arisen since this manuscript first appeared as a preprint. Drupieski [Dr1, Dr2] has used the methods of this paper to investigate the cohomology and representation theory for Frobenius-Lusztig kernels of quantum groups. Moreover, Feldvoss and Witherspoon have shown how to apply our calculations to determine the representation type of the small quantum group [FeW].

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## CHAPTER 1

# Preliminaries and Statement of Results

After introducing some notation, this chapter discusses the main results of this work. These results are all concerned with the cohomology algebra  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  of the small quantum group  $u_\zeta(\mathfrak{g})$  at a root  $\zeta$  of unity associated to a complex, simple Lie algebra  $\mathfrak{g}$ . Applications are presented to the calculation of support varieties over  $u_\zeta$  for certain classes of modules for the quantum enveloping algebra  $U_\zeta$ .

### 1.1. Some preliminary notation

Let  $\Phi$  be a finite and irreducible root system (in the classical sense). Fix a set  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  of simple roots (always labeled in the standard way [Bo, Appendix]), and let  $\Phi^+$  (respectively,  $\Phi^-$ ) be the corresponding set of positive (respectively, negative) roots. The set  $\Phi$  spans a real vector space  $\mathbb{E}$  with positive definite inner product  $\langle u, v \rangle$ ,  $u, v \in \mathbb{E}$ , adjusted so that  $\langle \alpha, \alpha \rangle = 2$  if  $\alpha \in \Phi$  is a short root. If  $\Phi$  has only one root length, all roots in  $\Phi$  are both “short” and “long”. Thus, if  $\Phi$  has two root lengths and if  $\alpha \in \Phi$  is a long root, then  $\langle \alpha, \alpha \rangle = 4$  (respectively, 6) when  $\Phi$  has type  $B_n, C_n, F_4$  (respectively,  $G_2$ ). For  $\alpha \in \Phi$ , put  $d_\alpha := \frac{\langle \alpha, \alpha \rangle}{2} \in \{1, 2, 3\}$ .

Write  $Q = Q(\Phi) := \mathbb{Z}\Phi$  (the root lattice) and  $Q^+ = Q^+(\Phi) := \mathbb{N}\Phi^+$  (the positive root cone). If  $J \subseteq \Pi$ , let  $\Phi_J = \Phi \cap \mathbb{Z}J$ . If  $J = \emptyset$ , put  $\Phi_J = \emptyset$ . If  $J \neq \emptyset$ ,  $\Phi_J$  is a closed subroot system of  $\Phi$ . By definition, a nonempty subset  $\Phi' \subseteq \Phi$  is a closed subroot system provided that, given  $\alpha \in \Phi'$ , it holds that  $-\alpha \in \Phi'$ , and, given  $\alpha, \beta \in \Phi'$  such that  $\alpha + \beta \in \Phi$ , it holds that  $\alpha + \beta \in \Phi'$ . As discussed below, for many of the closed subsystems  $\Phi'$  important in this work, it is possible to find  $w \in W$  and  $J \subseteq \Pi$  so that  $w(\Phi') = \Phi_J$ ; in other words, a simple set of roots for  $\Phi'$  can be extended to a simple set of roots for  $\Phi$ . If  $\alpha = \sum_{i=1}^n m_i \alpha_i \in \Phi$ , its height is defined to be  $\text{ht } \alpha := m_1 + \dots + m_n$ .

For  $\alpha \in \Phi$ , write  $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$  for the corresponding coroot. Therefore,  $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$  is the dual root system in  $\mathbb{E}$  defined by  $\Phi$ . We will always denote the short root of maximal height in  $\Phi$  by  $\alpha_0$ ; thus,  $\alpha_0^\vee$  is the unique long root of maximal length in  $\Phi^\vee$ . For  $1 \leq i, j \leq n$ , put  $c_{i,j} = \langle \alpha_j, \alpha_i^\vee \rangle \in \mathbb{Z}$ . The Cartan matrix  $C := [c_{i,j}]$  of  $\Phi$  is symmetrizable in the sense that  $DC$  is a symmetric matrix, letting  $D$  be the diagonal matrix  $\text{diag}[d_{\alpha_1}, \dots, d_{\alpha_n}]$ .

Let  $X_+ \subset \mathbb{E}$  be the positive cone of dominant weights, i.e.,  $X_+$  consists of all  $\varpi \in \mathbb{E}$  satisfying  $\langle \varpi, \alpha_i^\vee \rangle \in \mathbb{N}$ ,  $i = 1, \dots, n$ . Define the fundamental dominant weights  $\varpi_1, \dots, \varpi_n \in X_+$  by the condition that  $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ , for  $1 \leq i, j \leq n$ . Therefore,  $X_+ = \mathbb{N}\varpi_1 \oplus \dots \oplus \mathbb{N}\varpi_n$ . For convenience, we will occasionally let  $\varpi_0$  denote the 0-weight. Put  $X = \mathbb{Z}\varpi_1 \oplus \dots \oplus \mathbb{Z}\varpi_n$  (the weight lattice). For  $\varpi \in X$ ,  $\alpha \in \Phi$ ,  $\langle \varpi, \alpha \rangle = d_\alpha \langle \varpi, \alpha^\vee \rangle \in \mathbb{Z}$ . The weight lattice  $X$  is partially ordered by putting  $\lambda \geq \mu$  if and only if  $\lambda - \mu \in Q^+$ .

The Weyl group  $W$  of  $\Phi$  is the finite group of orthogonal transformations of  $\mathbb{E}$  generated by the reflections  $s_\alpha : \mathbb{E} \rightarrow \mathbb{E}$ ,  $\alpha \in \Phi$ , defined by  $s_\alpha(u) = u - \langle u, \alpha^\vee \rangle \alpha$ ,  $u \in \mathbb{E}$ . If  $S := \{s_{\alpha_1}, \dots, s_{\alpha_n}\}$ , then  $(W, S)$  is a Coxeter system. Let  $\ell : W \rightarrow \mathbb{N}$  be the length function on  $W$ ; thus, if  $w \in W$ ,  $\ell(w)$  is the smallest integer  $m$  such that  $w = s_{\beta_1} \cdots s_{\beta_m}$  for  $\beta_i \in \Pi$ .

Let  $l$  be a fixed positive integer. For  $\alpha \in \Phi, m \in \mathbb{Z}$ , let  $s_{\alpha, m} : \mathbb{E} \rightarrow \mathbb{E}$  be the affine transformation defined by

$$s_{\alpha, m}(u) = u - (\langle u, \alpha^\vee \rangle - ml)\alpha, \quad \forall u \in \mathbb{E}.$$

If  $m = 0$ , then  $s_{\alpha, m} = s_{\alpha, 0} = s_\alpha \in W$ . The affine Weyl group  $W_l$  is the subgroup of the group  $\text{Aff}(\mathbb{E})$  of affine transformations of  $\mathbb{E}$  generated by the reflections  $s_{\alpha, m}$ ,  $\alpha \in \Phi, m \in \mathbb{Z}$ . Let  $S_l := \{s_{\alpha_1}, \dots, s_{\alpha_n}, s_{\alpha_0, -1}\}$ . Then  $(W_l, S_l)$  is a Coxeter system. If  $lQ$  is identified as the subgroup of  $\text{Aff}(\mathbb{E})$  consisting of translations by  $l$ -multiples of elements in  $Q$ , then  $W_l \cong W \ltimes lQ$ . The extended affine Weyl group  $\widetilde{W}_l$  is obtained by putting  $\widetilde{W}_l = W \ltimes lX$ . Although  $\widetilde{W}_l$  need not be a Coxeter group, it contains  $W_l$  as a normal subgroup satisfying  $\widetilde{W}_l/W_l \cong X/Q$ . The quotient map  $\widetilde{W}_l \rightarrow W \cong \widetilde{W}_l/lX$  is denoted by  $w \mapsto \overline{w}$ .

We will generally use the “dot” action of  $W_l$  on  $\mathbb{E}$ : for  $w \in W_l, u \in \mathbb{E}$ , put  $w \cdot u := w(u + \rho) - \rho$ , where  $\rho := \varpi_1 + \dots + \varpi_n = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in X_+$  is the Weyl weight.

The Coxeter number of  $\Phi$  is defined to be  $h = \langle \rho, \alpha_0^\vee \rangle + 1 = \text{ht}(\alpha_0^\vee) + 1$ . Thus,  $h - 1$  is the height of the maximal root in  $\Phi^\vee$ , or, equivalently, in  $\Phi$ . The integer  $h$  plays an important role in this paper, and often serves as a kind of “event horizon” in representation theory.

Suppose that  $A$  and  $B$  are augmented algebras (over some common field). In case  $A$  is a subalgebra of  $B$ , it is called *normal* in  $B$  provided that  $BA_+ = A_+B$ , where  $A_+$  denotes the augmentation ideal of  $A$ . In this situation, we form the augmented algebra  $B//A := B/I$ , where  $I := BA_+$ , a two-sided ideal in  $B$ . It will sometimes be convenient to write  $A \trianglelefteq B$  to indicate that  $A$  is a normal subalgebra of  $B$ . If  $A$  is a normal subalgebra in  $B$ , there is, of course, a spectral sequence which relates the cohomology of  $B$  with that of  $A$  and  $A//B$ . It will play an important role later in this paper. See Lemma 2.8.1 for more details.

## 1.2. Main results

Let  $G = G_{\mathbb{C}}$  be the connected, simple, simply connected algebraic group over  $\mathbb{C}$  with Lie algebra  $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$  and root system  $\Phi$  with respect to a fixed maximal torus  $T \subset G$ . Let  $\mathfrak{t} = \text{Lie } T$  be the corresponding maximal toral subalgebra of  $\mathfrak{g}$ . Given  $\alpha \in \Phi$ , let  $\mathfrak{g}_\alpha \subset \mathfrak{g}$  be the  $\alpha$ -root space. Put  $\mathfrak{b}^+ = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$  (the positive Borel subalgebra of  $\mathfrak{g}$ ), and  $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$  (the opposite Borel subalgebra).

For  $J \subseteq \Pi$ , let  $\mathfrak{l}_J = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha$  be the Levi subalgebra containing  $\mathfrak{t}$  and having root system  $\Phi_J$ . Then  $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J \supseteq \mathfrak{b}$  is a parabolic subalgebra of  $\mathfrak{g}$ , where  $\mathfrak{u}_J = \bigoplus_{\alpha \in \Phi - \Phi_J} \mathfrak{g}_\alpha$  is the nilpotent radical of  $\mathfrak{p}_J$ . The parabolic subgroup of  $G$  with Lie algebra  $\mathfrak{p}_J$  is denoted  $P_J$ . The Levi factor of  $P_J$  having Lie algebra  $\mathfrak{l}_J$  is denoted  $L_J$ . In particular,  $B := P_J$  where  $J = \emptyset$  is the standard Borel subgroup of  $G$  containing  $T$  (as its Levi factor) and corresponding to the negative roots.

The group  $G$  acts on its Lie algebra  $\mathfrak{g}$  via the adjoint action. Under this action,  $G$  stabilizes the subset  $\mathcal{N} = \mathcal{N}(\mathfrak{g})$  of nilpotent elements in  $\mathfrak{g}$ —recall that  $x \in \mathfrak{g}$  is nilpotent provided that  $d\phi(x)$  acts as a nilpotent operator for any finite dimensional faithful rational representation  $\phi : G \rightarrow \text{GL}(V)$ . We will refer to  $\mathcal{N}$  as the nullcone of  $\mathfrak{g}$ . For  $J \subseteq \Pi$ , define

$$\mathcal{N}(\Phi_J) := G \cdot \mathfrak{u}_J \subseteq \mathcal{N} \subset \mathfrak{g}.$$

Because the quotient variety  $G/P_J$  is complete, an elementary geometric argument [St, p. 68] establishes that  $\mathcal{N}(\Phi_J)$  is a closed subvariety of  $\mathfrak{g}$ , which is clearly irreducible (since  $G$  is irreducible). Also,  $\mathcal{N}(\Phi_J)$  is  $G$ -stable under the restriction of the adjoint action of  $G$ ; it has codimension in  $\mathfrak{g}$  equal to  $\dim \mathfrak{l}_J$ . In particular,  $\mathcal{N}(\Phi_\emptyset) = \mathcal{N}$  is a closed, irreducible (and normal) subvariety of  $\mathfrak{g}$  of codimension  $n = \text{rank}(\mathfrak{g})$ .

The varieties  $\mathcal{N}(\Phi_J)$  play a central role in this paper (for particular subsets  $J \subseteq \Pi$  which depend on a choice of an integer  $l$  introduced below). They can also be described in another way. It is known that the parabolic subgroup  $P_J$  with Lie algebra  $\mathfrak{p}_J$  has an open, dense orbit in  $\mathfrak{u}_J$ . Choose any

element  $x$  belonging to this orbit, and let  $\mathcal{O} = \mathcal{C}_J$  be the  $G$ -orbit of  $x$ . Then  $\mathcal{N}(\Phi_J)$  is the Zariski closure  $\bar{\mathcal{O}}$  in  $\mathfrak{g}$  of  $\mathcal{O}$ . The orbits  $\mathcal{O}$  in  $\mathcal{N}$  which arise as some  $\mathcal{C}_J$ ,  $J \subseteq \Pi$ , are called Richardson orbits. If  $J = \emptyset$ , then  $\mathcal{N}(\Phi_J) = \mathcal{N}$  is the closure of the unique regular orbit in  $\mathcal{N}$ . If the Levi factors  $L_J$  and  $L_K$  are  $G$ -conjugate (i.e., if the subsets  $J, K$  of  $\Pi$  are  $W$ -conjugate), then [JR, Thm. 2.8] establishes that  $\mathcal{C}_J = \mathcal{C}_K$ . Thus,

$$(1.2.1) \quad \text{if } J, K \subseteq \Pi \text{ are } W\text{-conjugate, then } \mathcal{N}(\Phi_J) = \mathcal{N}(\Phi_K).$$

Let  $l > 1$  be a fixed positive integer. Let  $\zeta \in \mathbb{C}$  be a primitive  $l$ th root of unity, and let  $U_\zeta := U_\zeta(\mathfrak{g})$  be the (Lusztig) quantum enveloping algebra attached to the complex simple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$ . This algebra is obtained from a certain integral form of the generic quantum group over  $\mathbb{Q}(q)$  by specializing  $q$  to  $\zeta$ . In addition,  $U_\zeta$  has a Hopf algebra structure, which will be more fully discussed in Section 2.2 below. Furthermore,  $U_\zeta$  has a subalgebra  $u_\zeta = u_\zeta(\mathfrak{g})$  which is often called the “small” quantum group. It has finite dimension equal to  $l^{\dim \mathfrak{g}}$ . As an augmented subalgebra of  $U_\zeta$ , the small quantum group  $u_\zeta$  is normal. In fact,  $U_\zeta/u_\zeta \cong \mathbb{U}(\mathfrak{g})$ , where  $\mathbb{U}(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$ .

It will usually be necessary to impose some mild restrictions on the integer  $l$ . These restrictions are indicated in the two assumptions below. Throughout we will be careful to indicate when the restrictions are in force.

**Assumption 1.2.1.** *The integer  $l$  is odd and greater than 1. If the root system  $\Phi$  has type  $G_2$ , then 3 does not divide  $l$ . In addition,  $l$  is not a bad prime for  $\Phi$ . (See Section 3.1 for the definition of a bad prime.)*

Let

$$(1.2.2) \quad \Phi_0 = \Phi_{0,l} := \{\alpha \in \Phi \mid \langle \rho, \alpha^\vee \rangle \equiv 0 \pmod{l}\}.$$

If  $l$  satisfies Assumption 1.2.1, then  $\Phi_0$  is either the empty set  $\emptyset$  or a closed subroot system of  $\Phi$ . When  $\Phi_0$  is not empty, there exists a non-empty subset  $J \subseteq \Pi$  and an element  $w \in W$  such that

$$w(\Phi_0) = \Phi_J, \text{ and, moreover, such that } w(\Phi_0^+) = \Phi_J^+.$$

See Theorem 3.4.1. If  $w' \in W$  and  $J' \subseteq \Pi$  also satisfy  $w'(\Phi_0) = \Phi_{J'}$ , then (1.2.1) guarantees that  $\mathcal{N}(\Phi_0) = \mathcal{N}(\Phi_{J'})$ . Therefore, we can write  $\mathcal{N}(\Phi_0) := \mathcal{N}(\Phi_J)$  as a well-defined closed subvariety of the nullcone  $\mathcal{N}$ .

If  $l \geq h$ , then  $\Phi_0 = \emptyset$ . If  $l < h$ , an explicit subset  $J \subseteq \Pi$  such that  $\Phi_0$  is a  $W$ -conjugate of  $\Phi_J$  is given in Chapter 3 for each of the classical types  $A, B, C, D$ . Similar information for the exceptional types  $E_6, E_7, E_8, F_4, G_2$  (together with an explicit  $w \in W$  such that  $w(\Phi_0) = \Phi_J$ ) is collected in Appendix A.1.

We will often require the following additional restriction on  $l$ .

**Assumption 1.2.2.** *Let  $l$  be a fixed positive integer satisfying the Assumption 1.2.1. Moreover, assume that when the root system  $\Phi$  is of type  $B_n$  or type  $C_n$ , then  $l > 3$ .*

The theorem below presents a computation of the cohomology algebra  $H^\bullet(u_\zeta, \mathbb{C}) = \text{Ext}_{u_\zeta}^\bullet(\mathbb{C}, \mathbb{C})$  of the small quantum group  $u_\zeta$ . It is a well-known result (see [Mac, p. 232] or [GK, Cor. 5.4]) that, given any Hopf algebra  $H$  over a field  $k$ , the cohomology algebra  $H^\bullet(H, k)$  is a graded-commutative algebra (so that, in particular, the subalgebra concentrated in even degrees is a commutative algebra over  $k$ ). Additionally, because the algebras  $u_\zeta$  are normal subalgebras of  $U_\zeta$ , the cohomology spaces  $H^i(u_\zeta, \mathbb{C})$ , for any non-negative integer  $i$ , acquire a natural action of  $U_\zeta/u_\zeta \cong \mathbb{U}(\mathfrak{g})$ , the universal enveloping algebra of  $\mathfrak{g}$ ; see [GK, Lem. 5.2.1]. Since  $u_\zeta$  is finite dimensional, each space  $H^i(u_\zeta, \mathbb{C})$  is therefore a finite dimensional  $\mathfrak{g}$ -module, and thus it is a finite dimensional rational  $G$ -module. In the

following result, we identify  $H^\bullet(u_\zeta, \mathbb{C})$  as a rational  $G$ -module. In fact, in most cases, we can identify it as a rational  $G$ -algebra.

This theorem extends the main result due to Ginzburg and Kumar [GK, Main Thm.], which calculated the cohomology algebra  $H^\bullet(u_\zeta, \mathbb{C})$  in the special case in which  $l > h$ . Specifically, their work established that  $H^\bullet(u_\zeta, \mathbb{C}) = H^{2\bullet}(u_\zeta, \mathbb{C})$  is concentrated in even degrees, and it is isomorphic to the coordinate algebra  $\mathbb{C}[\mathcal{N}]$  of the nullcone  $\mathcal{N}$  of  $\mathfrak{g}$ .

**Theorem 1.2.3.** *Let  $l$  be as in Assumption 1.2.1 and let  $u_\zeta = u_\zeta(\mathfrak{g})$  be the small quantum group associated to the complex simple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$  at a primitive  $l$ th root of unity  $\zeta$ . Choose  $w \in W$  and  $J \subseteq \Pi$  such that  $w(\Phi_0^+) = \Phi_J^+$ . Assume that  $J$  is as listed in Chapter 3 (for the classical cases) and in Appendix A.1 (for the exceptional cases). Then the following identifications hold as  $G$ -modules under Assumption 1.2.1. Furthermore, in part(b)(i), the identification is as rational  $G$ -algebras under the additional Assumption 1.2.2.*

- (a) *The odd degree cohomology vanishes (i.e.,  $H^{2\bullet+1}(u_\zeta, \mathbb{C}) = 0$ ).*
- (b) *The even degree cohomology algebra  $H^{2\bullet}(u_\zeta, \mathbb{C})$  is described below.*
  - (i) *Suppose that  $l \nmid n+1$  when  $\Phi$  is of type  $A_n$  and  $l \neq 9$  when  $\Phi$  is of type  $E_6$ . Then*

$$H^{2\bullet}(u_\zeta, \mathbb{C}) \cong \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \times^{P_J} \mathfrak{u}_J]$$

*where  $G \times^{P_J} \mathfrak{u}_J$  is defined in Section 3.7. If we assume further that  $l \neq 7, 9$  when  $\Phi$  is of type  $E_8$ , then*

$$H^{2\bullet}(u_\zeta, \mathbb{C}) \cong \mathbb{C}[\mathcal{N}(\Phi_0)].$$

- (ii) *If  $\Phi$  is of type  $A_n$  and  $l \mid n+1$  with  $n+1 = l(m+1)$ , then*

$$H^{2\bullet}(u_\zeta, \mathbb{C}) \cong \text{ind}_{P_J}^G \left( \bigoplus_{t=0}^{l-1} S^{\frac{2\bullet - (m+1)t(l-t)}{2}}(\mathfrak{u}_J^*) \otimes \varpi_{t(m+1)} \right).$$

*(Recall the convention that  $\varpi_0 = 0$ .)*

- (iii) *If  $\Phi$  is of type  $E_6$  and  $l = 9$ , one can take  $J = \{\alpha_4\}$ . Then*

$$H^{2\bullet}(u_\zeta, \mathbb{C}) \cong \text{ind}_{P_J}^G \left( S^\bullet(\mathfrak{u}_J^*) \oplus (S^{\frac{2\bullet-12}{2}}(\mathfrak{u}_J^*) \otimes \varpi_1) \oplus (S^{\frac{2\bullet-12}{2}}(\mathfrak{u}_J^*) \otimes \varpi_6) \right).$$

We emphasize that the isomorphisms in (b(ii)) and (b(iii)) are only isomorphisms as rational  $G$ -modules. After considerable preparation in Chapters 3 and 4, Theorem 1.2.3 will be proved in Chapter 5 (see Sections 5.5 and 5.7).

In most cases in Theorem 1.2.3, the algebra  $H^\bullet(u_\zeta, \mathbb{C}) = H^{2\bullet}(u_\zeta, \mathbb{C})$  identifies with the coordinate algebra  $\mathbb{C}[\mathcal{N}(\Phi_0)]$  of the affine variety  $\mathcal{N}(\Phi_0)$ , and it is, therefore, a finitely generated algebra over  $\mathbb{C}$ . However, in Chapter 6, we show that, more generally, the algebra  $H^\bullet(u_\zeta, \mathbb{C})$  is a finitely generated  $\mathbb{C}$ -algebra. These remaining cases require individual arguments. More specifically, we have the following result.

**Theorem 1.2.4.** *Let  $l$  be as in Assumption 1.2.2.*

- (a) *The algebra  $H^\bullet(u_\zeta, \mathbb{C}) = H^{2\bullet}(u_\zeta, \mathbb{C})$  is a finitely generated, commutative  $\mathbb{C}$ -algebra.*
- (b) *For any finite dimensional  $u_\zeta$ -module  $M$ ,  $H^\bullet(u_\zeta, M)$  is finitely generated as a module for  $H^\bullet(u_\zeta, \mathbb{C})$  (where the action is described in [PW, Rem. 5.3, Appendix]).*

Theorem 1.2.4 has also recently been proven by Mastnak, Pevtsova, Schauenberg, and Witherspoon [MPSW, Corollary 6.5] only assuming the conditions of Assumption 1.2.1. They work in a more general context of pointed Hopf algebras.

The above theorem points to an important application to the theory of support varieties for  $u_\zeta$ . Namely, Theorem 1.2.4 implies that, given a finite dimensional  $u_\zeta$ -module  $M$ , the support variety

$\mathcal{V}_{\mathfrak{g}}(M)$  can be defined as the maximal ideal spectrum  $\text{Maxspec}(R/J_M)$ , letting  $J_M$  be the annihilator in  $R := H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  for its natural action on  $\text{Ext}_{u_\zeta(\mathfrak{g})}^\bullet(M, M)$  (cf. [PW, §5]).

For  $\lambda \in X_+$ , let  $\nabla_\zeta(\lambda) = H_\zeta^0(\lambda)$  denote the associated induced  $U_\zeta$ -module. (See Section 2.10 for more details on these modules—often called costandard modules.) Of course, each  $\nabla_\zeta(\lambda)$  can be regarded by restriction as a module for the small quantum group  $u_\zeta$ . Our third major result provides a computation of the support varieties of these modules.

**Theorem 1.2.5.** *Let  $l$  be as in Assumption 1.2.2. Let  $\lambda \in X_+$  and let  $\nabla_\zeta(\lambda) = H_\zeta^0(\lambda)$  be the costandard module for  $U_\zeta$  of high weight  $\lambda$ . Suppose that  $(l, p) = 1$  for any bad prime  $p$  of  $\Phi$ . Then there exists a subset  $J \subseteq \Pi$  such that*

$$\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = G \cdot \mathbf{u}_J.$$

Besides the costandard  $U_\zeta$ -modules, another important class of  $U_\zeta$ -modules are the Weyl (or standard) modules  $\Delta_\zeta(\lambda)$ ,  $\lambda \in X_+$ , which can be defined as duals of the costandard modules  $\nabla_\zeta(\lambda)$ . Precisely,  $\Delta_\zeta(\lambda) = \nabla_\zeta(\lambda^*)^*$ , where  $\lambda^*$  is the image of  $\lambda$  under the opposition involution on  $X_+$ . The modules  $\nabla_\zeta(\lambda)$  and  $\Delta_\zeta(\lambda)$  are both finite dimensional with characters given by Weyl's classical character formula. The statement of Theorem 1.2.5 remains valid if each  $\nabla_\zeta(\lambda)$  is replaced by  $\Delta_\zeta(\lambda)$ .

Theorem 1.2.5 is proved in Section 8.3, where the subset  $J$  is explicitly identified in terms of the closed subset  $\Phi_{\lambda, l}$  of  $\Phi$  (defined in (3.1.1)). In particular, the theorem shows that the support variety of  $\nabla_\zeta(\lambda)$  is the closure of a Richardson orbit in  $\mathcal{N}(\mathfrak{g})$ . We also calculate support varieties of  $\nabla_\zeta(\lambda)$  in the case of bad  $l$ . Within these computations we present examples, showing that even when  $l > h$ , if  $p \mid l$  where  $p$  is a bad prime for  $\Phi$ , then  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  need not be the closure of a Richardson orbit. The calculation of support varieties given in Theorem 1.2.5 perfectly mirrors similar results in [NPV] for the restricted Lie algebra of a simple algebraic group over a field of positive characteristic.



## CHAPTER 2

# Quantum Groups, Actions, and Cohomology

This section lays out the framework for the results in the paper. Even though our main results involve the computation of the cohomology for the small quantum group  $u_\zeta(\mathfrak{g})$ , it will be necessary to construct subalgebras  $\mathbb{U}_q(\mathfrak{u}_J)$  (and  $U_\zeta(\mathfrak{u}_J)$ ) corresponding to nilpotent radicals  $\mathfrak{u}_J$  of parabolic subalgebras  $\mathfrak{p}_J$  of  $\mathfrak{g}$ . This construction will necessitate the use of Lusztig's construction of a PBW basis for the full quantum enveloping algebras  $\mathbb{U}_q(\mathfrak{g})$ .

We also present results on the adjoint action of the  $\mathcal{A}$ -form (where  $\mathcal{A} = \mathbb{Q}[q, q^{-1}]$ ) of the quantum enveloping algebra  $\mathbb{U}_q(\mathfrak{p}_J)$  of the subalgebras  $\mathfrak{p}_J$  on the  $\mathcal{A}$ -form on the algebras  $\mathbb{U}_q(\mathfrak{u}_J)$  associated to the unipotent radical  $\mathfrak{u}_J$ . These results will be essential for our cohomological calculations.

We will also make use of the DeConcini-Kac quantum enveloping algebra  $\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g})$ , as well as its subalgebras  $\mathcal{U}_\zeta(\mathfrak{p}_J)$  and  $\mathcal{U}_\zeta(\mathfrak{u}_J)$ . While the (Lusztig) quantum enveloping algebra  $U_\zeta(\mathfrak{g})$  contains the small quantum group  $u_\zeta(\mathfrak{g})$  as a subalgebra,  $u_\zeta(\mathfrak{g})$  is a quotient algebra of  $\mathcal{U}_\zeta(\mathfrak{g})$ . In the final part of this section, we prove several results on the formal characters of cohomology groups of the algebras  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  in the context of an Euler characteristic calculation.

### 2.1. Listings

Let  $\Phi$  be an arbitrary root system with associated Weyl group  $W$ . Let  $w_0 \in W$  be the longest word, and set  $N = |\Phi^+|$ . A reduced expression  $w_0 = s_{\beta_1} \cdots s_{\beta_N}$  (where the  $\beta_i \in \Pi$  are not necessarily distinct) determines a listing

$$\gamma_1 = \beta_1, \gamma_2 = s_{\beta_1}(\beta_2), \dots, \gamma_N = s_{\beta_1} \cdots s_{\beta_{N-1}}(\beta_N)$$

of  $\Phi^+$ . Define a linear ordering on  $\Phi^+$  by setting, for  $\beta, \delta \in \Phi^+$ ,

$$\beta \prec \delta \iff \beta = \gamma_i, \delta = \gamma_j \text{ with } i < j.$$

Of course, this ordering depends on the reduced expression chosen for  $w_0$ .

Now let  $J \subseteq \Pi$ , and put  $\Phi_J = \Phi \cap \mathbb{Z}J$ , the subroot system  $\Phi_J$  generated by  $J$ . Set  $W_J = \langle s_\alpha : \alpha \in J \rangle$ , the Weyl group of  $\Phi_J$ , let  $w_{0,J} \in W_J$  be the longest word in  $W_J$ , and let  $w_J := w_{0,J}w_0$ . Observe that  $w_0 = w_{0,J}w_J$  satisfies  $\ell(w_0) = \ell(w_{0,J}) + \ell(w_J)$ . Once the subset  $J$  is fixed, it will often be useful to fix a reduced expression for  $w_0$  by first choosing one for  $w_{0,J}$  and then extending it by a reduced expression for  $w_J$ . If  $|\Phi_J^+| = M$ , then  $\gamma_1 \prec \cdots \prec \gamma_M$  lists the positive roots  $\Phi_J^+$ , while  $\gamma_{M+1} \prec \cdots \prec \gamma_N$  lists the remaining positive roots (those in  $\Phi^+ \setminus \Phi_J^+$ ). As above, this ordering depends upon the choice of reduced expressions.

Since  $w_{0,J}$  is the longest word in  $W_J$ , for  $\beta \in J$ ,

$$\ell(w_{0,J}s_\beta) < \ell(w_{0,J}).$$

Furthermore,  $w_J$  is a distinguished right coset representative for  $W_J$  in  $W$  in the sense that  $w_J$  is the unique element of minimal length in its coset  $W_J w_J$ . Thus,

$$(2.1.1) \quad \ell(s_\beta w_J) > \ell(w_J) \quad \text{and hence} \quad \ell(s_\beta w_J) = \ell(w_J) + 1.$$



## 2.2. Quantum enveloping algebras

We now assume that  $\Phi$  is irreducible, and make use of the notation introduced in Section 1.1. Let  $\mathcal{A} = \mathbb{Q}[q, q^{-1}]$  be the  $\mathbb{Q}$ -algebra of Laurent polynomials in an indeterminate  $q$ . Then  $\mathcal{A}$  has fraction field  $\mathbb{Q}(q)$ . We continue to restrict the integer  $l > 1$  according to Assumption 1.2.1. Let  $\zeta = \sqrt[l]{1} \in \mathbb{C}$  be a primitive  $l$ th root of unity and let  $k = \mathbb{Q}(\zeta) \subset \mathbb{C}$  be the cyclotomic field generated by  $\zeta$  over  $\mathbb{Q}$ . We will regard  $k$  as an  $\mathcal{A}$ -algebra by means of the homomorphism  $\mathbb{Q}[q, q^{-1}] \rightarrow k$ ,  $q \mapsto \zeta$ .

The quantum enveloping algebra  $\mathbb{U}_q = \mathbb{U}_q(\mathfrak{g})$  of  $\mathfrak{g}$  is the  $\mathbb{Q}(q)$ -algebra with generators  $E_\alpha, K_\alpha^{\pm 1}, F_\alpha$ ,  $\alpha \in \Pi$ , which are subject to the relations (R1)–(R6) listed in [Jan2, (4.3)] (whose notation we will generally follow, unless otherwise indicated). For example, the elements  $K_\alpha = K_\alpha^{+1}$  and  $K_\alpha^{-1}$ ,  $\alpha \in \Pi$ , generate a commutative subalgebra of  $\mathbb{U}_q$  such that  $K_\alpha K_\alpha^{-1} = 1$ . Also, for  $\alpha, \beta \in \Pi$ , we have

$$\begin{cases} K_\alpha E_\beta K_\alpha^{-1} = q^{\langle \beta, \alpha^\vee \rangle} E_\beta = q^{\langle \beta, \alpha \rangle} E_\beta; \\ K_\alpha F_\beta K_\alpha^{-1} = q_\alpha^{-\langle \beta, \alpha^\vee \rangle} F_\beta = q^{-\langle \beta, \alpha \rangle} F_\beta, \end{cases}$$

where  $q_\alpha = q^{d_\alpha}$ .

In addition, there exists a Hopf algebra structure on  $\mathbb{U}_q$ , with comultiplication  $\Delta : \mathbb{U}_q \rightarrow \mathbb{U}_q \otimes \mathbb{U}_q$  and antipode  $S : \mathbb{U}_q \rightarrow \mathbb{U}_q$  explicitly defined on generators by

$$(2.2.1) \quad \begin{cases} \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha \\ \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha \\ \Delta(K_\alpha^{\pm 1}) = K_\alpha^{\pm 1} \otimes K_\alpha^{\pm 1} \end{cases} \quad \text{and} \quad \begin{cases} S(E_\alpha) = -K_\alpha^{-1} E_\alpha \\ S(F_\alpha) = -F_\alpha K_\alpha \\ S(K_\alpha^{\pm 1}) = K_\alpha^{\mp 1}. \end{cases}$$

The comultiplication  $\Delta$  is an algebra homomorphism. Moreover, the antipode  $S$  is an algebra anti-isomorphism. The counit  $\epsilon : \mathbb{U}_q \rightarrow \mathbb{Q}(q)$  is the unique algebra homomorphism satisfying  $\epsilon(E_\alpha) = \epsilon(F_\alpha) = 0$  and  $\epsilon(K_\alpha^{\pm 1}) = 1$ . See [Jan2, Ch. 4] for more details.

Let  $\mathbb{U}_q^+$  denote the subalgebra of  $\mathbb{U}_q$  generated by the  $E_\alpha$  ( $\alpha \in \Pi$ ),  $\mathbb{U}_q^-$  denote the subalgebra of  $\mathbb{U}_q$  generated by the  $F_\alpha$  ( $\alpha \in \Pi$ ), and  $\mathbb{U}_q^0$  denote the subalgebra of  $\mathbb{U}_q$  generated by the  $K_\alpha^{\pm 1}$  ( $\alpha \in \Pi$ ). The generators of the algebra  $\mathbb{U}_q^+$  (resp.,  $\mathbb{U}_q^-$ ) are subject only to the Serre relations (R5) (resp., (R6)) in [Jan2, p. 53].

Multiplication gives isomorphisms of vector spaces

$$(2.2.2) \quad \mathbb{U}_q^+ \otimes \mathbb{U}_q^0 \otimes \mathbb{U}_q^- \xrightarrow{\sim} \mathbb{U}_q \xleftarrow{\sim} \mathbb{U}_q^- \otimes \mathbb{U}_q^0 \otimes \mathbb{U}_q^+.$$

There will be analogous subalgebras  $U_\zeta^+$ ,  $U_\zeta^-$  and  $U_\zeta^0$  of the algebra  $U_\zeta$  defined below upon specialization to  $\zeta$ . In this case, the maps analogous to those in (2.2.2) defined by algebra multiplication are also isomorphisms of vector spaces.

The quantum enveloping algebra  $\mathbb{U}_q$  has two  $\mathcal{A}$ -forms,  $\mathbb{U}_q^{\mathcal{A}}$  (due to Lusztig) and  $\mathcal{U}_q^{\mathcal{A}}$  (due to De Concini and Kac). In other words,  $\mathbb{U}_q^{\mathcal{A}}$  (resp.,  $\mathcal{U}_q^{\mathcal{A}}$ ) is an  $\mathcal{A}$ -subalgebra of  $\mathbb{U}_q$  which is free as an  $\mathcal{A}$ -module and which satisfies

$$\mathbb{U}_q^{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(q) \cong \mathbb{U}_q \cong \mathcal{U}_q^{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(q).$$

After passage to  $\mathbb{C}$ , these algebras play roles analogous to the hyperalgebra of a reductive group (over a field of positive characteristic) and the universal enveloping algebra of its Lie algebra, respectively. These  $\mathcal{A}$ -forms are defined below.

For an integer  $i$ , put

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}},$$

and set, for  $i > 0$ ,  $[i]^! = [i][i-1] \cdots [1]$ . By convention,  $[0]^! = 1$ . For any integer  $n$  and positive integer  $m$ , write

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n][n-1] \cdots [n-m+1]}{[1][2] \cdots [m]}.$$

Set  $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ , by definition. The expressions  $[i]$  and  $\begin{bmatrix} n \\ m \end{bmatrix}$  all belong to  $\mathcal{A}$  (in fact, they belong to  $\mathbb{Z}[q, q^{-1}]$ ). In case the root system has two root lengths, some scaling of the variable  $q$  is required. Thus, given any Laurent polynomial  $f \in \mathcal{A}$  and  $\alpha \in \Pi$ , let  $f_\alpha \in \mathcal{A}$  be obtained by replacing  $q$  throughout by  $q_\alpha = q^{d_\alpha}$ .

For  $\alpha \in \Pi$  and  $m \geq 0$ , let

$$\begin{cases} E_\alpha^{(m)} = \frac{E_\alpha^m}{[m]_\alpha!} \in \mathbb{U}_q \\ F_\alpha^{(m)} = \frac{F_\alpha^m}{[m]_\alpha!} \in \mathbb{U}_q. \end{cases}$$

be the  $m$ th “divided powers.” Let

$$\mathbb{U}_q^{\mathcal{A}} = \mathbb{U}_q^{\mathcal{A}}(\mathfrak{g}) := \langle E_\alpha^{(m)}, F_\alpha^{(m)}, K_\alpha^{\pm 1} \mid \alpha \in \Pi, m \in \mathbb{N} \rangle \subset \mathbb{U}_q,$$

where  $\langle \cdots \rangle$  means “ $\mathcal{A}$ -subalgebra generated by.” The algebra  $\mathbb{U}_q^{\mathcal{A}}$  admits a triangular factorization induced from (2.2.2), in which  $\mathbb{U}_q^+$  (respectively,  $\mathbb{U}_q^-, \mathbb{U}_q^0$ ) is replaced by the subalgebra  $\mathbb{U}_q^{\mathcal{A},+}$  (respectively,  $\mathbb{U}_q^{\mathcal{A},-}, \mathbb{U}_q^{\mathcal{A},0}$ ). Here  $\mathbb{U}_q^{\mathcal{A},+}$  (respectively,  $\mathbb{U}_q^{\mathcal{A},-}$ ) is defined to be the  $\mathcal{A}$ -subalgebra of  $\mathbb{U}_q^{\mathcal{A}}$  generated by the elements  $E_\alpha^{(m)}$  (respectively,  $F_\alpha^{(m)}$ ) for  $\alpha \in \Pi$  and  $m \in \mathbb{N}$ , and  $\mathbb{U}_q^{\mathcal{A},0}$  is defined to be the  $\mathcal{A}$ -subalgebra generated by the elements

$$K_\alpha^{\pm 1}, \begin{bmatrix} K_\alpha; m \\ n \end{bmatrix} := \prod_{i=1}^n \frac{K_\alpha q_\alpha^{m-i+1} - K_\alpha^{-1} q_\alpha^{-m-i+1}}{q_\alpha^i - q_\alpha^{-i}} \in \mathbb{U}_q^0,$$

for  $\alpha \in \Pi$ ,  $n \in \mathbb{N}$ , and  $m \in \mathbb{Z}$ .

Put

$$U_\zeta = U_\zeta(\mathfrak{g}) := \mathbb{U}_q^{\mathcal{A}} / \langle K_\alpha^l - 1, \alpha \in \Pi \rangle \otimes_{\mathcal{A}} \mathbb{C},$$

where in this expression  $\langle \cdots \rangle$  means “ideal generated by  $\cdots$ ”. Here  $\mathbb{C}$  is regarded as a  $\mathcal{A}$ -algebra via the algebra homomorphism  $\mathcal{A} \rightarrow \mathbb{C}$ ,  $q \mapsto \zeta$ . Let  $U_\zeta^+$  (respectively,  $U_\zeta^-, U_\zeta^0$ ) be the image in  $U_\zeta$  of  $\mathbb{U}_q^{\mathcal{A},+} \otimes_{\mathcal{A}} \mathbb{C}$  (respectively,  $\mathbb{U}_q^{\mathcal{A},-} \otimes_{\mathcal{A}} \mathbb{C}$ ,  $\mathbb{U}_q^{\mathcal{A},0} \otimes_{\mathcal{A}} \mathbb{C}$ ).

The Hopf algebra structure on  $\mathbb{U}_q$  induces a Hopf algebra structure on  $\mathbb{U}_q^{\mathcal{A}}$ , and then passage to the field  $\mathbb{C}$ , one obtains a Hopf algebra structure on the algebra  $U_\zeta$ . By abuse of notation, let  $E_\alpha^{(m)}$ , etc.,  $\alpha \in \Pi$ , also denote the corresponding elements  $1 \otimes E_\alpha^{(m)}$ , etc. in  $U_\zeta$  (note that  $E_\alpha^{(1)} = E_\alpha$ ). Therefore,  $E_\alpha^l = F_\alpha^l = 0$  and  $K_\alpha^l = 1$  (by construction) in  $U_\zeta$ . The elements  $E_\alpha, K_\alpha, F_\alpha$ ,  $\alpha \in \Pi$ , generate a finite dimensional Hopf subalgebra, denoted  $u_\zeta = u_\zeta(\mathfrak{g})$ , of  $U_\zeta$ . The algebra  $u_\zeta$  is often called the small quantum group.

We now consider the DeConcini-Kac quantum enveloping algebra  $\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g})$ . To begin, define  $\mathcal{U}_q^{\mathcal{A}}$  to be the  $\mathcal{A}$ -subalgebra of  $\mathbb{U}_q$  generated by the  $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ . There is an inclusion of  $\mathcal{A}$ -forms:  $\mathcal{U}_q^{\mathcal{A}} \subseteq \mathbb{U}_q^{\mathcal{A}}$ . Then set

$$\mathcal{U}_\mathbb{C} = \mathcal{U}_\mathbb{C}(\mathfrak{g}) := \mathcal{U}_q^{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C},$$

the algebra obtained by base-change (i. e.,  $\mathbb{C}$  is regarded as an  $\mathcal{A}$ -algebra via the same algebra homomorphism  $\mathcal{A} \rightarrow \mathbb{C}$ ,  $q \mapsto \zeta$ , as above.) Finally, put

$$\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g}) := \mathcal{U}_\mathbb{C} / \langle 1 \otimes K_\alpha^l - 1 \otimes 1, \alpha \in \Pi \rangle.$$

As with the quantum enveloping algebra  $U_\zeta$ , the algebra  $\mathcal{U}_\zeta$  also inherits a Hopf algebra structure from that of  $\mathbb{U}_q$ . However, unlike the inclusion  $\mathcal{U}_q^{\mathcal{A}} \subseteq \mathbb{U}_q^{\mathcal{A}}$  of  $\mathcal{A}$ -forms above,  $\mathcal{U}_\zeta$  is not generally a subalgebra

of  $U_\zeta$ . The algebra  $\mathcal{U}_\zeta$  has a central (and hence normal) subalgebra  $\mathcal{Z}$  such that  $u_\zeta \cong \mathcal{U}_\zeta // \mathcal{Z}$ ; see [DK, Section 3] for more details. An explicit description of  $\mathcal{Z}$  will be given in Section 2.7.

All the  $U_\zeta$ -modules  $M$  considered in this paper will be assumed to be integrable and type 1. Usually, this will be assumed without mention, but occasionally we repeat it for emphasis. In other words, each  $E_i, F_j$  acts locally nilpotently on  $M$ . In addition,  $M$  decomposes into a direct sum  $\bigoplus_{\lambda \in X} M_\lambda$  of  $M_\lambda$  weight spaces for  $\lambda \in X$ . Thus, if  $v \in M_\lambda$ ,

$$\begin{cases} K_\alpha v = \zeta^{\langle \lambda, \alpha \rangle} v; \\ [K_\alpha; m] v = [\langle \lambda, \alpha \rangle + m]_{q=\zeta^{d_\alpha}} v \end{cases}$$

for all  $\alpha \in \Pi, m \in \mathbb{Z}, n \in \mathbb{N}$ .

### 2.3. Connections with algebraic groups

Let  $\mathbb{U}(\mathfrak{g})$  be the classical universal enveloping algebra of the complex simple Lie algebra  $\mathfrak{g}$  with root system  $\Phi$ . The Lie algebra  $\mathfrak{g}$  has Chevalley generators  $e_\alpha, f_\alpha, h_\alpha, \alpha \in \Pi$ , which satisfy certain well-known relations, providing a presentation of the algebra  $\mathbb{U}(\mathfrak{g})$  [Hum, §18.3]. The Frobenius morphism is a surjective (Hopf) algebra homomorphism

$$\text{Fr} : U_\zeta \twoheadrightarrow \mathbb{U}(\mathfrak{g}).$$

If  $I$  is the augmentation ideal of  $u_\zeta$ , then  $\text{Ker Fr} = IU_\zeta = U_\zeta I$ . In other words,  $u_\zeta$  is a normal Hopf subalgebra of  $U_\zeta$  such that  $U_\zeta // u_\zeta \cong \mathbb{U}(\mathfrak{g})$ . Given  $\alpha \in \Pi$ , let  $m$  be a positive integer. If  $m = a\ell + b$ , where  $0 \leq b < \ell$ , then

$$\begin{cases} \text{Fr}(E_\alpha^{(m)}) = \delta_{b,0} e_\alpha^{(a)} \\ \text{Fr}(F_\alpha^{(m)}) = \delta_{b,0} f_\alpha^{(a)} \\ \text{Fr}([K_\alpha; \ell]) = h_\alpha. \end{cases}$$

In this expression, we have put  $e_\alpha^{(a)} = \frac{1}{a!} e_\alpha^a$  and  $f_\alpha^{(a)} = \frac{1}{a!} f_\alpha^a$  (cf. [L1]).

By means of the algebra homomorphism  $\text{Fr}$ , modules for the Lie algebra  $\mathfrak{g}$  can be “pulled back” to give modules for  $U_\zeta$ : given a  $\mathfrak{g}$ -module  $M$ ,  $M^{[1]} := \text{Fr}^* M$  denotes the  $U_\zeta$  module obtained from  $M$  by making  $x \in U_\zeta$  act as a linear transformation on  $M$  by  $\text{Fr}(x)$ . If  $M$  is a locally finite  $\mathbb{U}(\mathfrak{g})$ -module (e. g., finite dimensional), then  $M^{[1]}$  is an integrable, type 1  $U_\zeta$ -module. Conversely, a  $U_\zeta$ -module  $N$  on which  $u_\zeta$  acts trivially can be made into a  $\mathbb{U}(\mathfrak{g})$ -module via the Frobenius map, so that  $N \cong M^{[1]}$  for a locally finite  $\mathbb{U}(\mathfrak{g})$ -module  $M$ .

Let  $G$  be the complex simple, simply connected algebraic group having Lie algebra  $\mathfrak{g}$ . Let  $B \supset T$  (respectively,  $B^+ \supset T$ ) be the Borel subgroup corresponding to the set  $-\Phi^+ = \Phi^-$  (respectively,  $\Phi^+$ ) of negative (respectively, positive) roots. The category of locally finite  $\mathbb{U}(\mathfrak{g})$ -modules is naturally isomorphic to the category of rational  $G$ -modules.

The set  $X_+$  of dominant weights for the root system  $\Phi$  indexes the irreducible modules for  $U_\zeta$ . Given  $\lambda \in X_+$ , let  $L_\zeta(\lambda)$  be the irreducible  $U_\zeta$ -module of high weight  $\lambda$ . On the other hand, let  $L(\lambda)$  be the irreducible rational  $G$ -module of high weight  $\lambda \in X_+$ . We may also identify  $L(\lambda)$  with the finite dimensional irreducible  $\mathbb{U}(\mathfrak{g})$ -module of high weight  $\lambda$ . (Both  $L_\zeta(\lambda)$  and  $L(\lambda)$  are determined up to isomorphism.) For  $\lambda \in X_+$ , we have

$$\text{Fr}(L(\lambda)) = L(\lambda)^{[1]} \cong L_\zeta(\ell\lambda).$$

The integral group algebra  $\mathbb{Z}X$  has basis denoted  $e(\mu)$ ,  $\mu \in X$ . Given a finite dimensional  $U_\zeta$ -module  $M$ ,

$$\text{ch } M = \sum_{\mu \in X} \dim M_\mu e(\mu) \in \mathbb{Z}X$$

denotes its (formal) character, where  $M_\mu$  is the  $\mu$ -weight space for the action of  $U_\zeta^0$  on  $M$ . Sometimes for emphasis, we write  $\text{ch}_\zeta M$  for  $\text{ch } M$ .

Similarly, if  $M$  is a finite dimensional rational  $G$ -module (respectively,  $\mathbb{U}(\mathfrak{g})$ -module), let  $\text{ch } M \in \mathbb{Z}X$  be its formal character with respect to the fixed maximal torus  $T$  (respectively, Cartan subalgebra  $\mathfrak{h} = \text{Lie}(T)$ ).

## 2.4. Root vectors and PBW-basis

The “root vector” generators  $E_\alpha, F_\alpha$  for the quantum enveloping algebra  $\mathbb{U}_q$  are only defined for simple roots  $\alpha$ . In what follows, it will be necessary to work with root vectors defined for general (i.e., not simple) roots. In this direction, for each  $\alpha \in \Pi$ , Lusztig has defined an algebra automorphism  $T_\alpha : \mathbb{U}_q \rightarrow \mathbb{U}_q$ . Here we will follow [Jan2, Ch. 8], where the reader can find more details. If  $s = s_\alpha \in W$  is the simple reflection defined by  $\alpha$ , we often write  $T_s := T_\alpha$ . Given any  $w \in W$ , let  $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_n}$  be a reduced expression (so the  $\beta_i \in \Pi$ ). Then define  $T_w := T_{\beta_1} \cdots T_{\beta_n} \in \text{Aut}(\mathbb{U}_q)$ . The automorphism  $T_w$  is independent of the reduced expression of  $w$ . In other words, the automorphisms  $T_\alpha$  extend to an action of the braid group of  $W$  on  $\mathbb{U}_q$ .

Now let  $J \subseteq \Pi$  and fix a reduced expression  $w_0 = s_{\beta_1} \cdots s_{\beta_N}$  that begins with a reduced expression for the element  $w_{0,J}$  as in Section 2.1. If  $w_{0,J} = s_{\beta_1} \cdots s_{\beta_M}$ , then of course  $s_{\beta_{M+1}} \cdots s_{\beta_N}$  is a reduced expression for  $w_J = w_{0,J} w_0$ . There exists a linear ordering  $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_N$  of the positive roots, where  $\gamma_i = s_{\beta_1} \cdots s_{\beta_{i-1}}(\beta_i)$ . For  $\gamma = \gamma_i \in \Phi^+$ , the “root vector”  $E_\gamma \in \mathbb{U}_q^+$  is defined by

$$E_\gamma = E_{\gamma_i} := T_{s_{\beta_1} \cdots s_{\beta_{i-1}}}(E_{\beta_i}) = T_{\beta_1} \cdots T_{\beta_{i-1}}(E_{\beta_i}).$$

If  $\gamma$  is simple, the “new”  $E_\gamma$  agrees with the original generator  $E_\gamma$ . More generally,  $E_\gamma$  has weight  $\gamma$ . Similarly,

$$F_\gamma = F_{\gamma_i} := T_{s_{\beta_1} \cdots s_{\beta_{i-1}}}(F_{\beta_i}) = T_{\beta_1} \cdots T_{\beta_{i-1}}(F_{\beta_i}),$$

a root vector of weight  $-\gamma$ .

The subalgebra  $\mathbb{U}_q^+$  (respectively,  $\mathbb{U}_q^-$ ) has a PBW-like basis consisting of all monomials  $E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N}$  (respectively,  $F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N}$ ),  $a_1, \dots, a_N \in \mathbb{N}$ . Using divided powers when necessary, one obtains PBW-bases for the specialized quantum groups  $U_\zeta^+$ ,  $U_\zeta^-$ ,  $u_\zeta^+$ ,  $u_\zeta^-$ ,  $\mathcal{U}_\zeta^+$ , and  $\mathcal{U}_\zeta^-$ . The automorphisms  $T_w$  induce automorphisms on  $\mathbb{U}_q^A$  and hence on  $U_\zeta^A$ .

The monomial bases satisfy a key “commutativity” property originally observed by Levendorskiĭ and Soibelman. Here  $E_{\gamma_i}^0 = 1$  and  $F_{\gamma_i}^0 = 1$ .

**Lemma 2.4.1.** ([DP, Thm. 9.3], [LS]) *Let  $\mathcal{A}'$  be a subring of  $\mathbb{Q}(q)$  containing  $\mathcal{A}$ . Suppose that  $q^2 - q^{-2}$  is invertible in  $\mathcal{A}'$  when  $\Phi$  is of type  $B_n, C_n$ , or  $F_4$  and additionally  $q^3 - q^{-3}$  is invertible in  $\mathcal{A}'$  when  $\Phi$  is of type  $G_2$ . In  $\mathbb{U}_q$ , we have for  $i < j$*

- (a)  $E_{\gamma_i} E_{\gamma_j} = q^{\langle \gamma_i, \gamma_j \rangle} E_{\gamma_j} E_{\gamma_i} + (*)$  where  $(*)$  is an  $\mathcal{A}'$ -linear combination of monomials  $E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N}$ , with  $a_s = 0$  unless  $i < s < j$ ;
- (b)  $F_{\gamma_i} F_{\gamma_j} = q^{\langle \gamma_i, \gamma_j \rangle} F_{\gamma_j} F_{\gamma_i} + (*)$  where  $(*)$  is an  $\mathcal{A}'$ -linear combination of monomials  $F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N}$ , with  $a_s = 0$  unless  $i < s < j$ .

The requirements on  $\mathcal{A}'$  (i.e., that  $q^2 - q^{-2}$  is invertible in types  $B, C, F$ , etc.) are not explicitly stated in [DP], though they are implicit in the arguments in its appendix (see pp. 135–137 in [DP]; and also [Dr1, §3.2]). We thank Chris Drupieski for pointing this out to us.

## 2.5. Levi and parabolic subalgebras

Given a subset  $J \subseteq \Pi$ , consider the Levi and parabolic Lie subalgebras  $\mathfrak{l}_J$  and  $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$  of  $\mathfrak{g}$ . We denote the respective universal enveloping algebras by  $\mathbb{U}(\mathfrak{l}_J)$  and  $\mathbb{U}(\mathfrak{p}_J)$ . One can naturally define corresponding quantum enveloping algebras  $\mathbb{U}_q(\mathfrak{l}_J)$  and  $\mathbb{U}_q(\mathfrak{p}_J)$ . As (Hopf) subalgebras of  $\mathbb{U}_q$ ,

$\mathbb{U}_q(\mathfrak{l}_J)$  is the subalgebra generated by the elements  $\{E_\alpha, F_\alpha : \alpha \in J\} \cup \{K_\alpha^{\pm 1} : \alpha \in \Pi\}$ , and  $\mathbb{U}_q(\mathfrak{p}_J)$  is the subalgebra generated by  $\{E_\alpha : \alpha \in J\} \cup \{F_\alpha, K_\alpha^{\pm 1} : \alpha \in \Pi\}$ . For example, if  $J = \emptyset$ , then  $\mathfrak{l}_J = \mathfrak{h}$ ,  $\mathfrak{p}_J = \mathfrak{b}$ ,  $\mathbb{U}_q(\mathfrak{l}_J) = \mathbb{U}_q^0$ , and  $\mathbb{U}_q(\mathfrak{p}_J) = \mathbb{U}_q(\mathfrak{b}) = \mathbb{U}_q^- \cdot \mathbb{U}_q^0$ . Specializing gives the subalgebras  $U_\zeta(\mathfrak{l}_J)$ ,  $U_\zeta(\mathfrak{p}_J)$ ,  $u_\zeta(\mathfrak{l}_J)$ ,  $u_\zeta(\mathfrak{p}_J)$  of  $U_\zeta$ , and  $\mathcal{U}_\zeta(\mathfrak{l}_J)$  and  $\mathcal{U}_\zeta(\mathfrak{p}_J)$  of  $\mathcal{U}_\zeta$ .

If we denote by  $\mathfrak{p}_J^+ = \mathfrak{l}_J \oplus \mathfrak{u}_J^+$  the opposite parabolic subalgebra (containing the positive Borel subalgebra  $\mathfrak{b}^+$ ), then one can analogously consider  $\mathbb{U}_q(\mathfrak{p}_J^+)$ .

## 2.6. The subalgebra $\mathbb{U}_q(\mathfrak{u}_J)$

The purpose of this section is to define a subalgebra  $\mathbb{U}_q(\mathfrak{u}_J)$  of  $\mathbb{U}_q$ . It will play a role analogous to that played by the subalgebra  $\mathbb{U}(\mathfrak{u}_J)$  of the universal enveloping algebra  $\mathbb{U}(\mathfrak{p}_J)$  corresponding to the nilpotent radical of  $\mathfrak{p}_J$ . As above, choose a reduced expression for  $w_0$  (beginning with one for  $w_{0,J}$ ). Define  $\mathbb{U}_q(\mathfrak{u}_J)$  to be the subspace of  $\mathbb{U}_q^-$  spanned by the  $F_{\gamma_{M+1}}^{a_{M+1}} \cdots F_{\gamma_N}^{a_N}$ ,  $a_i \in \mathbb{N}$ . By Lemma 2.4.1,  $\mathbb{U}_q(\mathfrak{u}_J)$  is a subalgebra of  $\mathbb{U}_q^-$ . In addition, the monomials above form a basis for  $\mathbb{U}_q(\mathfrak{u}_J)$ . One can also verify directly that  $\mathbb{U}_q(\mathfrak{u}_J)$  is independent of the choice of reduced expression for  $w_0$ . However, this follows from a more general set-up which will be useful for other purposes as well.

Given any  $v \in W$ , we define as in [Jan2, 8.24] a subspace  $U^-[v] \subset \mathbb{U}_q^-$  (respectively,  $U^+[v] \subset \mathbb{U}_q^+$ ) as follows. Choose a reduced expression  $v = s_{\eta_1} s_{\eta_2} \cdots s_{\eta_t}$ . For  $1 \leq i \leq t$ , set  $f_i = T_{s_{\eta_1} s_{\eta_2} \cdots s_{\eta_{i-1}}}(F_{\eta_i})$  (respectively,  $e_i = T_{s_{\eta_1} s_{\eta_2} \cdots s_{\eta_{i-1}}}(E_{\eta_i})$ ). The  $f_i$  (respectively,  $e_i$ ) are in some sense “root vectors” like those defined earlier. Then  $U^-[v]$  (respectively,  $U^+[v]$ ) is defined to be the span of all monomials of the form  $f_1^{a_1} f_2^{a_2} \cdots f_t^{a_t}$  (respectively,  $e_1^{a_1} e_2^{a_2} \cdots e_t^{a_t}$ ) for  $a_i \geq 0$ . By [DKP, 2.2],  $U^-[v]$  (respectively,  $U^+[v]$ ) is a subalgebra of  $\mathbb{U}_q^-$  (resp.,  $\mathbb{U}_q^+$ ) and moreover is independent of the choice of reduced expression for  $v$ .

Since  $\mathbb{U}_q(\mathfrak{u}_J) = T_{w_{0,J}}(U^-[w_J])$ ,  $\mathbb{U}_q(\mathfrak{u}_J)$  is a subalgebra of  $\mathbb{U}_q(\mathfrak{p}_J) \subset \mathbb{U}_q$ , independent of the reduced expression for  $w_J$ . Furthermore, since the automorphism  $T_{w_{0,J}}$  is also independent of the choice of reduced expression for  $w_{0,J}$ , the algebra  $\mathbb{U}_q(\mathfrak{u}_J)$  depends only on  $J$ , not on our above choices of reduced expressions. Following the procedure in Section 2.2, the algebra  $\mathbb{U}_q(\mathfrak{u}_J)$  can be specialized to give algebras  $U_\zeta(\mathfrak{u}_J)$ ,  $u_\zeta(\mathfrak{u}_J)$ , and  $\mathcal{U}_\zeta(\mathfrak{u}_J)$ . For example,  $U_\zeta(\mathfrak{u}_J)$  is the subalgebra of  $U_\zeta^-$  spanned by  $F_{\gamma_{M+1}}^{(a_{M+1})} \cdots F_{\gamma_N}^{(a_N)}$ ,  $a_i \in \mathbb{N}$ .

Similarly one can define a subalgebra  $\mathbb{U}_q(\mathfrak{u}_J^+) \subset \mathbb{U}_q(\mathfrak{p}_J)$ , and the corresponding specializations. Then  $\mathbb{U}_q(\mathfrak{u}_J^+) = T_{w_{0,J}}(U^+[w_J])$ . When  $J = \emptyset$ ,  $\mathbb{U}_q(\mathfrak{u}_J) = \mathbb{U}_q^-$  and  $\mathbb{U}_q(\mathfrak{u}_J^+) = \mathbb{U}_q^+$ .

## 2.7. Adjoint action

Given a Hopf algebra  $A$ , the adjoint action of  $A$  is defined by setting, for  $x, y \in A$ ,  $\text{Ad}(x)(y) = \sum x_{(1)} y S(x_{(2)})$  where  $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$  is the comultiplication and  $S$  is the antipode. We consider, in particular, the case  $A = \mathbb{U}_q$ , where we record the formulas here on generators: (for  $m \in \mathbb{U}_q$ )

$$(2.7.1) \quad \begin{cases} \text{Ad}(E_\alpha)(m) &= E_\alpha m - K_\alpha m K_\alpha^{-1} E_\alpha \\ \text{Ad}(K_\alpha^{\pm 1})(m) &= K_\alpha^{\pm 1} m K_\alpha^{\mp 1} \\ \text{Ad}(F_\alpha)(m) &= (F_\alpha m - m F_\alpha) K_\alpha. \end{cases}$$

These expressions follow immediately from the formulas (2.2.1).

By twisting the Hopf structure on  $\mathbb{U}_q$  by the algebra involution  $\omega$  of  $\mathbb{U}_q$  given by  $\omega(E_\alpha) = F_\alpha$ ,  $\omega(F_\alpha) = E_\alpha$ , and  $\omega(K_\alpha) = K_\alpha^{-1}$ , we obtain another Hopf structure on  $\mathbb{U}_q$ , with comultiplication  ${}^\omega\Delta := (\omega \otimes \omega) \circ \Delta \circ \omega$  and antipode  ${}^\omega S = \omega \circ S \circ \omega$ ; see [Jan2, 3.8]. In particular, this procedure then

gives an alternate adjoint action  $\overline{\text{Ad}}$ , which satisfies:

$$(2.7.2) \quad \begin{cases} \overline{\text{Ad}}(E_\alpha)(m) &= (E_\alpha m - m E_\alpha) K_\alpha^{-1} \\ \overline{\text{Ad}}(K_\alpha^\pm 1)(m) &= K_\alpha^{\pm 1} m K_\alpha^\mp 1 \\ \overline{\text{Ad}}(F_\alpha)(m) &= F_\alpha m - K_\alpha^{-1} m K_\alpha F_\alpha. \end{cases}$$

**Proposition 2.7.1.** *The following stability results hold:*

- (a) *The subalgebra  $\mathbb{U}_q(\mathfrak{u}_J^+)$  is stable under the  $\text{Ad}$  action of  $\mathbb{U}_q(\mathfrak{p}_J^+)$  on itself.*
- (b) *The subalgebra  $\mathbb{U}_q(\mathfrak{u}_J)$  is stable under the  $\overline{\text{Ad}}$  action of  $\mathbb{U}_q(\mathfrak{p}_J)$  on itself.*

PROOF. Part (b) follows from part (a) since, for  $x \in \mathbb{U}_q(\mathfrak{p}_J)$ ,  $\overline{\text{Ad}}(x) = \omega \circ \text{Ad}(\omega(x)) \circ \omega$ .

We prove part (a). Since  $\mathbb{U}_q(\mathfrak{p}_J^+)$  is generated as an algebra by  $\{E_\alpha, K_\alpha, K_\alpha^{-1} : \alpha \in \Pi\} \cup \{F_\alpha : \alpha \in J\}$ , it suffices to show that  $\mathbb{U}_q(\mathfrak{u}_J^+)$  is stable under the action of these elements. By (2.7.1),  $\text{Ad}(K_\alpha)$  and  $\text{Ad}(K_\alpha^{-1})$  simply act by scalar multiplication on the weight spaces which span  $\mathbb{U}_q(\mathfrak{u}_J^+)$ . Hence  $\mathbb{U}_q(\mathfrak{u}_J^+)$  is stable under  $\text{Ad}(K_\alpha^\pm 1)$ . For  $\alpha \in \Phi^+ \setminus \Phi_J^+$ , the stability under  $\text{Ad}(E_\alpha)$  follows from the fact that  $\mathbb{U}_q(\mathfrak{u}_J^+)$  is an algebra. It remains to prove stability under  $\text{Ad}(F_\alpha)$  and  $\text{Ad}(E_\alpha)$  for  $\alpha \in J$ . Fix  $\alpha \in J$  and set  $\beta = -w_{0,J}(\alpha) \in J$ .

By Section 2.6,  $\mathbb{U}_q(\mathfrak{u}_J^+) = T_{w_{0,J}}(U^+[w_J])$ . Given a reduced expression  $w_J = s_{\eta_1} s_{\eta_2} \cdots s_{\eta_t}$ ,  $U^+[w_J]$  is spanned by monomials of ordered root vectors:

$$E_{\eta_1}, T_{s_{\eta_1}}(E_{\eta_2}), \dots, T_{s_{\eta_1} \cdots s_{\eta_{t-1}}}(E_{\eta_t}).$$

Since  $w_J = w_{0,J} w_0$ ,  $w_J^{-1}(\beta) \in \Pi$ . Consider also  $U^+[w_J s_{w_J^{-1}(\beta)}]$ . Since  $w_J s_{w_J^{-1}(\beta)} = s_\beta w_J$ , by (2.1.1),  $\ell(w_J s_{w_J^{-1}(\beta)}) = \ell(w_J) + 1$ . Therefore, the ordered root vectors defining  $U^+[w_J s_{w_J^{-1}(\beta)}]$  are

$$E_{\eta_1}, T_{s_{\eta_1}}(E_{\eta_2}), \dots, T_{s_{\eta_1} \cdots s_{\eta_{t-1}}}(E_{\eta_t}), T_{w_J}(E_{w_J^{-1}(\beta)}).$$

By [Jan2, Prop. 8.20]  $T_{w_J}(E_{w_J^{-1}(\beta)}) = E_{w_J(w_J^{-1}(\beta))} = E_\beta$ . Now  $U^+[w_J]$  is a subalgebra of  $U^+[w_J s_{w_J^{-1}(\beta)}]$  spanned by monomials not involving the last root vector  $E_\beta$ . Moreover, by Lemma 2.4.1, since  $E_\beta$  appears last in the ordering of root vectors in  $U^+[w_J s_{w_J^{-1}(\beta)}]$ ,

$$(2.7.3) \quad \text{if } u \in U^+[w_J] \subset U^+[w_J s_{w_J^{-1}(\beta)}] \text{ is a monomial, then } u E_\beta - q^{\langle \text{wt}(u), \beta \rangle} E_\beta u \in U^+[w_J],$$

where  $\text{wt}(u)$  denotes the weight of  $u$ .

Now consider the subalgebra  $U^+[s_\beta w_J]$ . The ordered root vectors defining  $U^+[s_\beta w_J]$  are

$$E_\beta, T_{s_\beta}(E_{\eta_1}), T_{s_\beta}(T_{s_{\eta_1}}(E_{\eta_2})), \dots, T_{s_\beta}(T_{s_{\eta_1} \cdots s_{\eta_{t-1}}}(E_{\eta_t})).$$

Then  $T_{s_\beta}(U^+[w_J])$  is evidently a subalgebra of  $U^+[s_\beta w_J]$  spanned by monomials in all but the first root vector  $E_\beta$ . Moreover, by Lemma 2.4.1, since  $E_\beta$  occurs first in the root ordering,

$$(2.7.4) \quad \text{if } u \in U^+[w_J] \text{ is a monomial, then } E_\beta T_{s_\beta}(u) - q^{\langle \text{wt}(T_{s_\beta}(u)), \beta \rangle} T_{s_\beta}(u) E_\beta \in T_{s_\beta}(U^+[w_J]).$$

We now show that  $\text{Ad}(F_\alpha)$  preserves  $\mathbb{U}_q(\mathfrak{u}_J^+)$ . Since  $\alpha, \beta \in \Pi$ , by [Jan2, Prop. 8.20],

$$F_\alpha = T_{w_{0,J} s_\beta}(F_\beta),$$

while [Jan2, 8.14(3)] gives that

$$F_\beta = -T_{s_\beta}(E_\beta) K_\beta^{-1} = -T_{s_\beta}(E_\beta K_\beta).$$

Combining these equalities gives

$$F_\alpha = T_{w_{0,J} s_\beta}(-T_{s_\beta}(E_\beta K_\beta)) = -T_{w_{0,J}}(E_\beta K_\beta).$$

Let  $T_{w_0,J}(u)$  for  $u \in U^+[w_J]$  be an arbitrary monomial element of  $\mathbb{U}_q(\mathfrak{u}_J^+)$ . Then we have

$$\begin{aligned}
\text{Ad}(F_\alpha)(T_{w_0,J}(u)) &= (F_\alpha T_{w_0,J}(u) - T_{w_0,J}(u) F_\alpha) K_\alpha && \text{by (2.7.1)} \\
&= (-T_{w_0,J}(E_\beta K_\beta) T_{w_0,J}(u) + T_{w_0,J}(u) T_{w_0,J}(E_\beta K_\beta)) K_\alpha && \text{from above} \\
&= T_{w_0,J}((u E_\beta K_\beta - E_\beta K_\beta u) K_\beta^{-1}) && [\text{Jan2, 8.18(3)}] \\
&= T_{w_0,J}(u E_\beta - E_\beta K_\beta u K_\beta^{-1}) \\
&= T_{w_0,J}(u E_\beta - q^{\langle \text{wt}(u), \beta \rangle} E_\beta u) \\
&\in T_{w_0,J}(U^+[w_J]) = \mathbb{U}_q(\mathfrak{u}_J^+) && \text{by (2.7.3)}
\end{aligned}$$

as claimed.

Lastly, we consider  $E_\alpha$ . Again  $\alpha, \beta \in \Pi$ , so  $E_\alpha = T_{w_0,J s_\beta}(E_\beta)$ . Let  $T_{w_0,J}(u)$  for  $u \in U^+[w_J]$  be an arbitrary monomial element of  $\mathbb{U}_q(\mathfrak{u}_J^+)$ . Then we have

$$\begin{aligned}
\text{Ad}(E_\alpha)(T_{w_0,J}(u)) &= E_\alpha T_{w_0,J}(u) - K_\alpha T_{w_0,J}(u) K_\alpha^{-1} E_\alpha && \text{by (2.7.1)} \\
&= T_{w_0,J s_\beta}(E_\beta) T_{w_0,J}(u) - K_\alpha T_{w_0,J}(u) K_\alpha^{-1} T_{w_0,J s_\beta}(E_\beta) && \text{from above} \\
&= T_{w_0,J s_\beta}(E_\beta T_{s_\beta}(u) - K_\beta T_{s_\beta}(u) K_\beta^{-1} E_\beta) && [\text{Jan2, 8.18(3)}] \\
&= T_{w_0,J s_\beta}(E_\beta T_{s_\beta}(u) - q^{\langle \text{wt}(T_{s_\beta}(u)), \beta \rangle} T_{s_\beta}(u) E_\beta) \\
&\in T_{w_0,J s_\beta}(T_{s_\beta}(U^+[w_J])) = T_{w_0,J}(U^+[w_J]) = \mathbb{U}_q(\mathfrak{u}_J^+) && \text{by (2.7.4)}
\end{aligned}$$

as claimed.  $\square$

The definitions of  $\text{Ad}$  and  $\overline{\text{Ad}}$  now give the following.

**Corollary 2.7.2.** *The algebra  $\mathbb{U}_q(\mathfrak{u}_J)$  is normal in  $\mathbb{U}_q(\mathfrak{p}_J)$  and the algebra  $\mathbb{U}_q(\mathfrak{u}_J^+)$  is normal in  $\mathbb{U}_q(\mathfrak{p}_J^+)$ . Furthermore, normality also holds for the specializations  $U_\zeta(\mathfrak{u}_J) \subset U_\zeta(\mathfrak{p}_J)$ ,  $u_\zeta(\mathfrak{u}_J) \subset u_\zeta(\mathfrak{p}_J)$ , and  $\mathcal{U}_\zeta(\mathfrak{u}_J) \subset \mathcal{U}_\zeta(\mathfrak{p}_J)$  (as well as for the  $+$ -versions). Also,  $U_\zeta(\mathfrak{l}_J) \cong U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{u}_J)$ ,  $u_\zeta(\mathfrak{l}_J) \cong u_\zeta(\mathfrak{p}_J)/u_\zeta(\mathfrak{u}_J)$ , and  $\mathcal{U}_\zeta(\mathfrak{l}_J) \cong \mathcal{U}_\zeta(\mathfrak{p}_J)/\mathcal{U}_\zeta(\mathfrak{u}_J)$ .*

Recall the inclusion of  $\mathcal{A}$ -forms  $\mathcal{U}_q^{\mathcal{A}} \subseteq \mathbb{U}_q^{\mathcal{A}}$  mentioned in §2.2. While it is generally false that  $\mathcal{U}_q^{\mathcal{A}}$  is stable under the adjoint action of  $\mathbb{U}_q^{\mathcal{A}}$  on itself, this property does hold after base-change from  $\mathcal{A}$  to a larger algebra  $\mathcal{B}$ . More precisely, given  $l$  satisfying Assumption 1.2.1, let  $f_l(x) \in \mathbb{Q}[x]$  denote the minimal polynomial for a primitive  $l$ th root of unity. Set  $\mathcal{B} := \mathbb{Z}[q, q^{-1}]_{\langle \langle f_l(q) \rangle \rangle}$ , i.e., the local ring determined by the maximal ideal  $\langle f_l(q) \rangle$ . Defining  $\mathbb{U}_q^{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathbb{U}_q^{\mathcal{A}}$  and  $\mathcal{U}_q^{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{U}_q^{\mathcal{A}}$ , there is an inclusion  $\mathcal{U}_q^{\mathcal{B}} \subset \mathbb{U}_q^{\mathcal{B}}$ .

**Lemma 2.7.3.** *[ABG, Prop. 2.9.2(i)] The adjoint action of  $\mathbb{U}_q^{\mathcal{B}}$  stabilizes  $\mathcal{U}_q^{\mathcal{B}}$ . That is,*

$$\text{Ad}(\mathbb{U}_q^{\mathcal{B}})(\mathcal{U}_q^{\mathcal{B}}) \subseteq \mathcal{U}_q^{\mathcal{B}} \quad \text{and} \quad \overline{\text{Ad}}(\mathbb{U}_q^{\mathcal{B}})(\mathcal{U}_q^{\mathcal{B}}) \subseteq \mathcal{U}_q^{\mathcal{B}}.$$

Hence, the adjoint action (either  $\text{Ad}$  or  $\overline{\text{Ad}}$ ) of  $U_\zeta$  defines an action on  $\mathcal{U}_\zeta$ .

Let  $\mathcal{Z} \subset \mathcal{U}_\zeta$  be the central subalgebra such that  $\mathcal{U}_\zeta/\mathcal{Z} \cong u_\zeta$  (cf. Section 2.2). In terms of root vectors,  $\mathcal{Z}$  is the algebra generated by the elements  $\{E_\gamma^l, F_\gamma^l : \gamma \in \Phi^+\}$ . Given  $J \subseteq \Pi$ , define a central subalgebra  $Z_J \subset \mathcal{U}_\zeta(\mathfrak{u}_J)$  as

$$Z_J := \mathcal{Z} \cap \mathcal{U}_\zeta(\mathfrak{u}_J).$$

Clearly  $Z_J$  is the subalgebra generated by  $\{F_\gamma^l : \gamma \in \Phi^+ \setminus \Phi_J^+\}$ . Further,  $\mathcal{U}_\zeta(\mathfrak{u}_J)/Z_J \cong u_\zeta(\mathfrak{u}_J)$ , leading to the following generalization of [ABG, Prop. 2.9.2].

**Corollary 2.7.4.** *Under the induced  $\overline{\text{Ad}}$ -action of  $U_\zeta$  on  $\mathcal{U}_\zeta$  we have the following:*

- (a)  $\overline{\text{Ad}}(U_\zeta(\mathfrak{p}_J))$  stabilizes  $\mathcal{U}_\zeta(\mathfrak{u}_J)$ ,  $Z_J$ , and  $u_\zeta(\mathfrak{u}_J)$ .
- (b) The action of  $u_\zeta(\mathfrak{p}_J)$  on  $Z_J$  is trivial.
- (c)  $\overline{\text{Ad}}$  induces an action of  $\mathbb{U}(\mathfrak{p}_J)$  on  $Z_J$ .

We conclude this section with some remarks on the action of a Hopf algebra  $H$  on an augmented algebra  $A$ . More precisely,  $A$  is defined to be a left (respectively, right)  $H$ -module algebra provided that  $A$  is a left (respectively, right)  $H$ -module such that:

- (i)  $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$  (respectively,  $(ab) \cdot h = \sum (a \cdot h_{(1)})(b \cdot h_{(2)})$ ) for  $a, b \in A, h \in H$ , putting  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ ;
- (ii)  $h \cdot 1_A = \epsilon(h)1_A$  (respectively,  $1_A \cdot h = \epsilon(h)1_A$ ) for  $h \in H$ ;
- (iii)  $\epsilon_A(h \cdot a) = \epsilon(h)\epsilon_A(a)$  (respectively,  $\epsilon_A(a \cdot h) = \epsilon_A(a)\epsilon(h)$ ), for  $a \in A, h \in H$ , where  $\epsilon_A$  is the augmentation map on  $A$ .

The notion of a left  $H$ -module algebra for a not necessarily augmented algebra  $A$  is studied further in [Mon, 4.1.1, 4.1.9].

For example,  $H$  is a right  $H$ -module algebra defined by

$$a \cdot h = \text{Ad}_r(h)(a) = \sum S(h_{(1)})ah_{(2)}, \quad a, h \in H.$$

Thus,  $\text{Ad}_r(hh') = \text{Ad}_r(h')\text{Ad}_r(h)$ ,  $h, h' \in H$ . In the context of  $\mathbb{U}_q$ , we have

$$(2.7.5) \quad \begin{cases} \text{Ad}_r(E_\alpha) = \overline{\text{Ad}}(K_\alpha^{-1}E_\alpha) \\ \text{Ad}_r(K_\alpha^{\pm 1}) = \overline{\text{Ad}}(K^{\mp 1}) \\ \text{Ad}_r(F_\alpha) = -\overline{\text{Ad}}(F_\alpha K_\alpha). \end{cases}$$

With these relations, there are evident analogues of the above lemma and its corollary. In particular, the augmented algebra  $\mathcal{U}_q$  is a left (respectively, right) module algebra for  $U_\zeta$  using  $\text{Ad}$  or  $\overline{\text{Ad}}$  (respectively,  $\text{Ad}_r$ ). Under the right action of  $U_\zeta$  on  $\mathcal{U}_\zeta$ , Corollary 2.7.4 holds replacing  $\overline{\text{Ad}}$  throughout by  $\text{Ad}_r$ .

Now let  $A$  be a right  $H$ -module algebra, and let  $V$  be a left  $A$ -module. A left action  $v \mapsto h \cdot v$  of  $H$  on  $V$  is called compatible with that of  $A$  provided that  $V$  is a left  $H$ -module with this action and  $a(h \cdot v) = \sum h_{(1)} \cdot (a \cdot h_{(2)})v$  for  $a \in A, v \in V, h \in H$ . If  $A$  is a left  $H$ -module algebra and  $V$  is a left  $H$ -module, the action is compatible provided the last condition is replaced by  $h \cdot (av) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot v)$ .

## 2.8. Finite dimensionality of cohomology groups

For any field  $k$ , given augmented  $k$ -algebras  $A \subset B$  with  $A$  normal in  $B$  (cf. Sections 1.1 and 2.7), there exists a Lyndon-Hochschild-Serre (LHS) spectral sequence (cf. [GK, §5.2, 5.3]). More precisely, we have the following result.<sup>1</sup>

**Lemma 2.8.1.** *Assume that  $B$  is flat as a right  $A$ -module. For any left  $B$ -module  $M$ , there is a natural action of the quotient algebra  $B//A$  on  $H^\bullet(A, M)$  which gives rise to a (first quadrant) spectral sequence*

$$E_2^{i,j} = H^i(B//A, H^j(A, M)) \Rightarrow H^{i+j}(B, M).$$

In this result, the action of  $B//A$  arises from the identification  $H^\bullet(A, M) \cong \text{Ext}_B^\bullet(B//A, M)$ . Then the right multiplication action of  $B//A$  on itself induces a left  $B//A$ -module structure on  $\text{Ext}_B^\bullet(B//A, M)$ . In some cases, the action of  $B//A$  also comes about through an action of  $B//A$  on the reduced bar resolution  $D_\bullet = D_\bullet(A)$  of  $k$ . More precisely, suppose that  $H$  is a Hopf algebra acting on the augmented algebra  $A$  on the *right* as explained above. Then  $H$  acts on the right on  $D_\bullet$ . Here  $D_n = A \otimes_k A_+^{\otimes n}$ ; explicitly,

$$a \otimes [a_1 | \cdots | a_n] \cdot h = \sum a \cdot h_{(1)}[a_1 \cdot h_{(2)} | \cdots | a_n \cdot h_{(n+1)}].$$

This action commutes with the differentials  $d_\bullet : D_n \rightarrow D_{n-1}$ . Now suppose that  $M$  is a left  $A$ -module with a compatible left action of  $H$ . The left action of  $H$  on  $M$  and the right action of  $H$  on  $B_\bullet$  define

<sup>1</sup>Recall that  $H^\bullet(A, M) := \text{Ext}_A^\bullet(k, M)$ .



a left  $H$ -action on the complex  $\text{Hom}_A(D_\bullet, M)$  computing  $H^\bullet(A, M)$ . The  $H$ -action on  $H^\bullet(A, M)$  can be interpreted in the context of Lemma 2.8.1, by putting  $B = H \# A$ , the smash product of  $H$  and  $A$ . (See [Mon].) Then  $B$  is a flat right  $A$ -module,  $M$  inherits a natural  $B$ -module structure and  $B//A \cong H$ . It is easy to verify that the two actions of  $H$  on  $H^\bullet(A, M)$  identify.<sup>2</sup>

We return to the situation of the previous section. Under the  $\overline{\text{Ad}}$ -action of  $U_\zeta(\mathfrak{p}_J)$  on itself,  $U_\zeta(\mathfrak{p}_J)$  admits the structure of a Hopf module algebra. The action of  $U_\zeta(\mathfrak{p}_J)$  on  $\mathcal{U}_\zeta(\mathfrak{p}_J)$  can be extended to an action on the bar resolution which computes the cohomology  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{p}_J), \mathbb{C})$ . Thus, there is a natural action of  $U_\zeta(\mathfrak{p}_J)$  on  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{p}_J), \mathbb{C})$  and further on  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ . In particular,  $U_\zeta^0$  acts on these cohomology spaces.

**Lemma 2.8.2.** *For each nonnegative integer  $n$ , the cohomology space  $H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$  is a finite dimensional vector space.*

PROOF. If  $Z_J$  is the central subalgebra of  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  defined after Lemma 2.7.3,  $\mathcal{U}_\zeta(\mathfrak{u}_J)//Z_J \cong u_\zeta(\mathfrak{u}_J)$ . Certainly,  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  is flat (even free) over  $Z_J$ . By Lemma 2.8.1, there is a spectral sequence

$$E_2^{i,j} = H^i(u_\zeta(\mathfrak{u}_J), H^j(Z_J, \mathbb{C})) \Rightarrow H^{i+j}(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}).$$

The subalgebra  $Z_J$  is central so it follows from [GK, Lemma 5.2.2] that the action of  $u_\zeta(\mathfrak{u}_J)$  on  $H^\bullet(Z_J, \mathbb{C})$  is trivial, and

$$E_2^{\bullet,\bullet} \cong H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes H^\bullet(Z_J, \mathbb{C}).$$

Since  $Z_J$  is the symmetric algebra based on the vector space  $\mathfrak{u}_J^{[1]}$ , we get that  $H^\bullet(Z_J, \mathbb{C}) \cong \Lambda^\bullet(\mathfrak{u}_J^*)^{[1]}$ . Moreover,  $u_\zeta(\mathfrak{u}_J)$  is finite dimensional, so  $H^i(u_\zeta(\mathfrak{u}_J), \mathbb{C})$  is finite dimensional for any integer  $i$ . The result follows because the cohomology  $H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$  is a subquotient of  $\oplus_{i+j=n} E_2^{i,j}$  which is finite dimensional.  $\square$

Also, the cohomology of  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  can be computed by means of the reduced bar resolution (cf. [Mac, Ch. X, §2]). Although it is not clear from this point of view that the cohomology is finite dimensional in any homological degree, it does have a natural  $U_\zeta^0$ -action induced from the  $\overline{\text{Ad}}$ -action on  $U_\zeta(\mathfrak{p}_J)$ . But then the above lemma establishes it is finite dimensional, and this fact implies each cohomology space is a weight module for  $U_\zeta^0$ . We summarize this result as follows.

**Corollary 2.8.3.** *The cohomology  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$  is a weight module for  $U_\zeta^0$ . For any  $\lambda \in X$ ,  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})_\lambda$  is finite dimensional.*

PROOF. It remains only to establish the last statement. But if  $\bar{D}_\bullet = D_\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J))$  is the reduced bar-resolution, then, given any weight  $\mu$ , for large  $n$ ,  $(\bar{D}_n)_\mu = 0$ .  $\square$

## 2.9. Spectral sequences and the Euler characteristic

We study the cohomology of  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  and  $u_\zeta(\mathfrak{u}_J)$ , using a filtration on  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  (respectively,  $u_\zeta(\mathfrak{u}_J)$ ) introduced for  $\mathcal{U}_\zeta$  in [DK].

Let  $A = \mathcal{U}_\zeta(\mathfrak{u}_J)$ . By Section 2.6,  $A$  has a basis of monomial elements

$$\{F_{\gamma_{M+1}}^{a_{M+1}} F_{\gamma_{M+2}}^{a_{M+2}} \cdots F_{\gamma_N}^{a_N} : a_i \in \mathbb{N}\},$$

where  $N = |\Phi^+|$  and  $M = |\Phi_J^+|$ . For  $\bar{a} := (a_{M+1}, \dots, a_N) \in \mathbb{N}^{N-M}$ , set

$$F_{\bar{a}} := F_{\gamma_{M+1}}^{a_{M+1}} F_{\gamma_{M+2}}^{a_{M+2}} \cdots F_{\gamma_N}^{a_N}.$$

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<sup>2</sup>We are grateful for discussions on these issues with Chris Drupieski, who pointed out that our original set-up with  $H$  acting on the left of  $A$  was not correct.

Place a total (lexicographical) ordering  $\prec$  on  $\mathbb{N}^{N-M}$  as follows. Put  $\bar{a} \prec \bar{b}$  if and only if there exists  $M+1 \leq i \leq N$  such that  $a_i < b_i$  and  $a_j = b_j$  for all  $j > i$ . With this ordering, one can define an  $\mathbb{N}^{N-M}$ -filtration on  $A$ . Given  $\bar{a} \in \mathbb{N}^{N-M}$ , let  $A_{\bar{a}}$  be the subalgebra of  $A$  generated by

$$\{F_{\bar{b}} : \bar{b} \preceq \bar{a}\}.$$

By Lemma 2.4.1, this is a multiplicative filtration on  $A$ . That is,  $A_{\bar{a}} \cdot A_{\bar{b}} \subseteq A_{\bar{a}+\bar{b}}$ .

Moreover, by Lemma 2.4.1 again, the associated graded algebra  $\text{gr } A$  is generated by  $\{X_{\alpha} : \alpha \in \Phi^+ \setminus \Phi_J^+\}$  subject to the relations:

$$X_{\alpha} \cdot X_{\beta} = \zeta^{(\alpha, \beta)} X_{\beta} \cdot X_{\alpha} \text{ if } \alpha \prec \beta.$$

This filtration induces a filtration on  $u_{\zeta}(\mathfrak{u}_J)$  such that the algebra  $\text{gr } u_{\zeta}(\mathfrak{u}_J)$  is also generated by  $\{X_{\alpha} : \alpha \in \Phi^+ \setminus \Phi_J^+\}$  subject to the above relations, in addition to the condition:

$$X_{\alpha}^l = 0 \text{ for } \alpha \in \Phi^+ \setminus \Phi_J^+.$$

Let  $\Lambda_{\zeta, J}^{\bullet}$  be the graded algebra with generators  $\{x_{\alpha} : \alpha \in \Phi^+ \setminus \Phi_J^+\}$  where  $\deg(x_{\alpha}) = 1$  subject to the following relations:

$$\begin{aligned} x_{\alpha} \cdot x_{\beta} + \zeta^{-(\alpha, \beta)} x_{\beta} \cdot x_{\alpha} &= 0 \text{ if } \alpha \prec \beta; \\ x_{\alpha}^2 &= 0 \text{ for } \alpha \in \Phi^+ \setminus \Phi_J^+. \end{aligned}$$

There exists a graded (by degree) algebra isomorphism  $H^{\bullet}(\text{gr } \mathcal{U}_{\zeta}(\mathfrak{u}_J), \mathbb{C}) \cong \Lambda_{\zeta, J}^{\bullet}$  (cf. [GK, Prop. 2.1]). This is also an isomorphism of  $U_{\zeta}^0$ -modules, where  $\Lambda_{\zeta, J}^{\bullet}$  is regarded as a  $U_{\zeta}^0$ -module by assigning  $x_{\alpha}$  weight  $\alpha$ .

**Proposition 2.9.1.** (a) *In the character group  $\mathbb{Z}X$ , we have*

$$\sum_{n=0}^{\infty} (-1)^n \text{ch } H^n(\mathcal{U}_{\zeta}(\mathfrak{u}_J), \mathbb{C}) = \sum_{n=0}^{\infty} (-1)^n \text{ch } \Lambda_{\zeta, J}^n.$$

(b) *If  $\lambda \in X$  is a weight of  $U_{\zeta}^0$  in  $H^n(\mathcal{U}_{\zeta}(\mathfrak{u}_J), \mathbb{C})$ , then  $\lambda$  is a weight of  $U_{\zeta}^0$  in  $\Lambda_{\zeta, J}^n$ .*

PROOF. Let  $A = \mathcal{U}_{\zeta}(\mathfrak{u}_J)$ . By Corollary 2.8.3 and the discussion above, both  $H^{\bullet}(A, \mathbb{C})$  and  $H^{\bullet}(\text{gr } A, \mathbb{C})$  have weight space decompositions with finite dimensional weight spaces. Let  $A_+$  and  $\text{gr } A_+$  denote the augmentation ideals of  $A$  and  $\text{gr } A$ , respectively. Let  $C^{\bullet}(A)$  and  $C^{\bullet}(\text{gr } A)$  be the complexes obtained by taking duals of the respective reduced bar resolutions. More precisely,  $C^n(A) = \text{Hom}_{\mathbb{C}}((A_+)^{\otimes n}, \mathbb{C})$  and  $C^n(\text{gr } A) = \text{Hom}_{\mathbb{C}}((\text{gr } A_+)^{\otimes n}, \mathbb{C})$ . Note that  $A_+$  and  $\text{gr } A_+$  are isomorphic as  $U_{\zeta}^0$ -modules. The same holds for  $C^n(A)$  and  $C^n(\text{gr } A)$  and the differentials of both complexes are  $U_{\zeta}^0$ -module maps. Thus, for a weight  $\lambda$ ,  $H^{\bullet}(A, \mathbb{C})_{\lambda}$  and  $H^{\bullet}(\text{gr } A, \mathbb{C})_{\lambda}$  identify with the cohomologies of the complexes  $C^{\bullet}(A)_{\lambda}$  and  $C^{\bullet}(\text{gr } A)_{\lambda}$ , respectively.

By the Euler-Poincaré principle (cf. [GW, Lemma 7.3.11]),

$$\begin{aligned} \chi(H^{\bullet}(A, \mathbb{C})_{\lambda}) &:= \sum_{n=0}^{\infty} (-1)^n \dim H^n(A, \mathbb{C})_{\lambda} \\ &= \sum_{n=0}^{\infty} (-1)^n \dim C^n(A)_{\lambda} \\ &= \sum_{n=0}^{\infty} (-1)^n \dim C^n(\text{gr } A)_{\lambda} \\ &= \sum_{n=0}^{\infty} (-1)^n \dim H^n(\text{gr } A, \mathbb{C})_{\lambda} \\ &=: \chi(H^{\bullet}(\text{gr } A, \mathbb{C})_{\lambda}). \end{aligned}$$

Part (a) follows from the cohomology calculation for  $\text{gr } A$  noted above.

Let  $A_\bullet$  be the increasing filtration on  $A$  indexed by  $\Lambda := \mathbb{N}^{N-M}$ , viewed as a poset using the lexicographic ordering  $\prec$  above. It induces a (downward) filtration on the complex  $C^\bullet(A)_\lambda$  as follows. For  $\gamma, \eta \in \Lambda$ , set  $A_{+\gamma} = A_\gamma \cap A_+$ , and define

$$\begin{cases} B_{[\prec\eta]}^n = \sum_{\sum \gamma_i \prec \eta} A_{+\gamma_1} \otimes A_{+\gamma_2} \otimes \cdots \otimes A_{+\gamma_n}, \\ B_{[\preceq\eta]}^n = \sum_{\sum \gamma_i \preceq \eta} A_{+\gamma_1} \otimes A_{+\gamma_2} \otimes \cdots \otimes A_{+\gamma_n}. \end{cases}$$

Then  $B_{[\prec\eta]}^n \subseteq B_{[\preceq\eta]}^n$ , so setting, for  $\lambda \in X(T)$ ,

$$\begin{cases} C^n(A)_{\lambda, [\prec\eta]} = \text{Hom}_{\mathbb{C}}(A_+^{\otimes n} / B_{[\prec\eta]}^n, \mathbb{C})_\lambda, \\ C^n(A)_{\lambda, [\preceq\eta]} = \text{Hom}_{\mathbb{C}}(A_+^{\otimes n} / B_{[\preceq\eta]}^n, \mathbb{C})_\lambda, \end{cases}$$

it follows that

$$C^n(A)_{\lambda, [\preceq\eta]} \subseteq C^n(A)_{\lambda, [\prec\eta]}.$$

Moreover, if  $\eta, \zeta \in \Lambda$  with  $\zeta \prec \eta$ , then  $C^n(A)_{\lambda, [\prec\eta]} \subseteq C^n(A)_{\lambda, [\prec\zeta]}$ .

The grading on  $\text{gr } A$  leads in a natural way to a grading of the complex  $C^\bullet(\text{gr } A)_\lambda$ . For  $\eta \in \Lambda$ ,  $C^\bullet(\text{gr } A)_{\lambda, [\eta]}$  denotes the graded component corresponding to  $\eta$ , and we can identify

$$C^\bullet(A)_{\lambda, [\prec\eta]} / C^\bullet(A)_{\lambda, [\preceq\eta]}$$

with  $C^\bullet(\text{gr } A)_{\lambda, [\eta]}$ . Also,

$$C^\bullet(\text{gr } A)_\lambda \cong \bigoplus_{\eta \in \Lambda} C^\bullet(A)_{\lambda, [\prec\eta]} / C^\bullet(A)_{\lambda, [\preceq\eta]}.$$

For a fixed weight  $\lambda$ ,  $C^n(\text{gr } A)_\lambda \neq 0$  for finitely many  $n$ , and  $C^\bullet(A)_{\lambda, [\prec\eta]} / C^\bullet(A)_{\lambda, [\preceq\eta]} \neq 0$  for finitely many  $\eta$ . Let  $\overline{\Lambda} = \{\eta \in \Lambda : C_{\lambda, [\prec\eta]}^n / C_{\lambda, [\preceq\eta]}^n \neq 0, \text{ for some } n\}$ . Then  $\overline{\Lambda}$  is a finite totally ordered set in  $\mathbb{N}^{N-M}$  which induces a filtration (which can be indexed by  $\mathbb{N}$ ) on  $C^\bullet(\text{gr } A)_\lambda$ . Therefore, we have a spectral sequence

$$E_1^{i,j} = (H^{i+j}(\text{gr } A, \mathbb{C})_{\lambda})_{(i)} \Rightarrow H^{i+j}(A, \mathbb{C})_\lambda.$$

This shows that  $H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})_\lambda$  is a subquotient of  $(\Lambda_{\zeta, J}^n)_\lambda$ , and part (b) follows.  $\square$

## 2.10. Induction functors

Let  $\mathcal{C}$  (respectively,  $\mathcal{C}^\leq$ ) be the category of type 1, integrable representations of  $U_\zeta$  (respectively,  $U_\zeta(\mathfrak{b})$ ). The restriction functor  $\text{res} : \mathcal{C} \rightarrow \mathcal{C}^\leq$  has a right adjoint induction functor  $H_\zeta^0(-) = H^0(U_\zeta/U_\zeta(\mathfrak{b}), -) : \mathcal{C}^\leq \rightarrow \mathcal{C}$  defined by

$$H_\zeta^0(M) = (M \otimes k[U_\zeta])^{U_\zeta(\mathfrak{b})} \cong \mathcal{F}(\text{Hom}_{U_\zeta(\mathfrak{b})}(U_\zeta, M)).$$

In this expression  $k[U_\zeta]$  denotes the coordinate algebra of  $U_\zeta$ . Also, the functor  $\mathcal{F}(-)$  assigns to any  $U_\zeta$ -module the largest type 1, integrable submodule. We refer to [APW, (2.8), (2.10)] and [RH, (2.9)] for further discussion and explanation of notation.

Any dominant weight  $\lambda \in X_+$  can be viewed as a one-dimensional  $U_\zeta(\mathfrak{b})$ -module, and so provides an induced module

$$\nabla_\zeta(\lambda) := H^0(U_\zeta/U_\zeta(\mathfrak{b}), \lambda).$$

This module has an irreducible socle isomorphic to  $L_\zeta(\lambda)$ . In addition, there is an equality

$$\text{ch } \nabla_\zeta(\lambda) = \text{ch } L(\lambda) = \sum_{x \in W} (-1)^{\ell(x)} e(w \cdot \lambda) / \sum_{x \in W} (-1)^{\ell(x)} e(x \cdot 0)$$

of formal characters, in which the expression on the right is just the Weyl character formula. (Recall that  $L(\lambda)$  denotes the irreducible representation for the complex group  $G$  (or its Lie algebra  $\mathfrak{g}$ ) of high weight  $\lambda$ .)

We will also use the induction functors  $H^0(U_\zeta/U_\zeta(\mathfrak{p}_J), -)$  (respectively,  $H^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), -)$ ) from the category of type 1, integrable  $U_\zeta(\mathfrak{p}_J)$ -modules (respectively,  $U_\zeta(\mathfrak{b})$ -modules) to type 1, integrable  $U_\zeta$ -modules (respectively,  $U_\zeta(\mathfrak{p}_J)$ -modules). Note that if  $\lambda$  is a one-dimensional  $U_\zeta(\mathfrak{b})$ -module then  $H^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), \lambda)$  is trivial as a  $U_\zeta(\mathfrak{u}_J)$ -module.

Let  $(X_J)_+ \subseteq X$  be the set of  $J$ -dominant weights, i.e.,  $\lambda \in X$  belongs to  $(X_J)_+$  if and only if  $\langle \lambda, \alpha^\vee \rangle \in \mathbb{N}$  for all  $\alpha \in J$ . The set  $(X_J)_+$  indexes the irreducible (type 1, integrable)  $U_\zeta(\mathfrak{l}_J)$ -modules. For  $\lambda \in (X_J)_+$ ,

$$\nabla_{J,\zeta}(\lambda) := H^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), \lambda)$$

has irreducible socle isomorphic to  $L_{J,\zeta}(\lambda)$ , the irreducible  $U_\zeta(\mathfrak{l}_J)$ -module of high weight  $\lambda$ .

If  $\lambda \in (X_J)_+$  satisfies  $\langle \lambda + \rho, \alpha^\vee \rangle = \ell - 1$  for all  $\alpha \in J$ , we call  $\lambda$  a  $J$ -Steinberg weight. Then  $\nabla_{J,\zeta}(\lambda)$  is a projective (and injective) irreducible  $U_\zeta(\mathfrak{l}_J)$ -module (in the category of type 1, integrable modules). It remains irreducible, projective, and injective upon restriction to  $\mathfrak{u}_\zeta(\mathfrak{l}_J)$ .

Finally, it will usually be more convenient to write  $\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)}(-)$  in place of  $H^0(U_\zeta(\mathfrak{p}_J)/U_\zeta(\mathfrak{b}), -)$ .



## CHAPTER 3

### Computation of $\Phi_0$ and $\mathcal{N}(\Phi_0)$

Throughout this chapter,  $\Phi$  is an irreducible root system in an Euclidean space  $\mathbb{E}$  with weight lattice  $X$ . We will be concerned with certain special closed subroot systems  $\Phi_{\lambda,l}$  of  $\Phi$  which are defined by an integer  $l > 1$  and a weight  $\lambda \in X$ . These are introduced in Section 3.1. After classifying these subroot systems in Sections 3.2 and 3.3 (see also Appendix A.1 for the exceptional cases), Section 3.5 takes up some related issues involving the normality of orbit closures. Finally, Section 3.6 uses these results to identify the coordinate algebras of these orbit closures as certain induced  $G$ -modules. All these results will play an important role later in this paper; see Chapter 4, for example.

#### 3.1. Subroot systems defined by weights

A prime  $p$  is called *bad* for the root system  $\Phi$  provided that there exists a closed subsystem  $\Phi'$  of  $\Phi$  such that  $Q/Q'$  has  $p$ -torsion, where  $Q = Q(\Phi)$  and  $Q' = Q(\Phi')$  are the root lattices of  $\Phi$  and  $\Phi'$ , respectively. If  $p$  is not bad, then  $p$  is called a *good* prime for  $\Phi$ . Equivalently,  $p$  is good if and only if  $p$  does not appear as the coefficient of a simple root in the decomposition of the maximal root in  $\Phi$  as an integral linear combination of simple roots; see [SS, I, §4]. The good primes for the various types of irreducible root systems are thus easily determined. Therefore, making use of the explicit expressions for the maximal root in  $\Phi$  displayed in [Bo, Plates I–IX], the good primes are given as follows:

- $\Phi$  of type  $A_n$ , all  $p$ ;
- $\Phi$  of type  $B_n, C_n, D_n$ ,  $p \geq 3$ ;
- $\Phi$  of type  $E_6, E_7, F_4, G_2$ ,  $p \geq 5$ ;
- $\Phi$  of type  $E_8$ ,  $p \geq 7$ .

Now let  $l > 1$  be an integer (not necessarily prime). We will say that  $l$  is good for  $\Phi$  provided that  $l$  is not divisible by a bad prime for  $\Phi$ . Otherwise,  $l$  is bad for  $\Phi$ .

Additionally, a good integer  $l$  is said to be *very good* for  $\Phi$  provided that if  $\Phi$  has type  $A_n$ , then  $l$  and  $n + 1$  are relatively prime. In cases of non-irreducible root systems (which may arise in the case of the root system of a Levi factor of a parabolic subgroup), the integer  $l$  is good (respectively, very good) provided that it is good (respectively, very good) for every irreducible component of  $\Phi$ .

The significance of good integers  $l$  comes about because of certain closed subsystems  $\Phi_{\lambda,l}$  constructed from weights  $\lambda \in X$ . Precisely, put

$$\Phi_{\lambda,l} := \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha \rangle \equiv 0 \pmod{l}\}.$$

Recall that, by our conventions, for  $\mu \in X$  and  $\alpha \in \Phi$ ,  $\langle \mu, \alpha \rangle \in \mathbb{Z}$ . In practice, when  $l$  is clear from context, we will just denote the subset  $\Phi_{\lambda,l}$  simply by  $\Phi_\lambda$ . Obviously,  $\Phi_\lambda$  (when it is not the empty set) is a closed subroot system of  $\Phi$ .

When working with a quantum enveloping algebra  $U_\zeta$ , where  $\zeta = \sqrt[l]{1}$ , our assumptions on  $l$  (i.e.,  $l$  is odd in types  $B_n, C_n$  and  $F_4$ , and, in type  $G_2$ ,  $l$  is not divisible by 3) mean that each integer  $d_\alpha = \frac{\langle \alpha, \alpha \rangle}{2}$  is relatively prime to  $l$ . Since  $d_\alpha \alpha^\vee = \alpha$ , it therefore follows that

$$(3.1.1) \quad \Phi_\lambda = \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \equiv 0 \pmod{l}\}.$$

Thus, in the sequel, we can always take (3.1.1) as the definition of  $\Phi_\lambda$ . We denote by  $\Phi_\lambda^+$  the intersection of  $\Phi_\lambda$  with  $\Phi^+$ . It is useful to observe that, for any  $w \in \widetilde{W}_l$ ,

$$\overline{w}(\Phi_\lambda) = \Phi_{w \cdot \lambda}.$$

The following result, while quite elementary, is essential for our work in this paper.

**Lemma 3.1.1.** *Let  $l > 1$  be an odd integer. Assume that  $l$  is good for  $\Phi$ . For  $\lambda \in X$ , there exists a set of simple roots  $\Pi'$  for  $\Phi$  such that  $\Pi' \cap \Phi_\lambda$  is a set of simple roots for  $\Phi_\lambda$ . In particular, there exists a  $w \in W$  and a subset  $J \subseteq \Pi$  such that  $w(\Phi_\lambda) = \Phi_J$ . Furthermore,  $w$  may be chosen so that  $w(\Phi_\lambda^+) = \Phi_J^+$ .*

PROOF. First, observe that

$$\mathbb{Q}\Phi_\lambda \cap \Phi = \Phi_\lambda.$$

In fact, if a root  $\alpha$  belongs to the left-hand side, then  $m\alpha \in \mathbb{Z}\Phi_\lambda$ , for some integer  $m$  which is divisible only by bad primes. Then  $\langle \lambda + \rho, m\alpha \rangle \equiv 0 \pmod{l}$ . But since  $(m, l) = 1$ , it follows that  $\langle \lambda + \rho, \alpha \rangle \equiv 0 \pmod{l}$ , i.e.,  $\alpha \in \Phi_\lambda$ . This means that the hypotheses of [Bo, IV.1.7, Prop. 24] are satisfied, and this quoted result implies the conclusion of the lemma. The final claim follows from the fact that all simple systems in  $\Phi_J$  are conjugate under the action of the Weyl group.  $\square$

Consider the case of  $\Phi_0$ . For all  $l \geq h$ , it is immediately true that  $\Phi_0 = \emptyset$ , and so the conclusion of the lemma trivially holds. However, this is not the case in general. For example, suppose that  $\Phi$  has type  $F_4$  with  $l = 3$ . Letting  $\epsilon_i$ ,  $1 \leq i \leq 4$ , be an orthonormal basis for  $\mathbb{R}^4$ ,  $\Phi$  can be realized explicitly as the following set of 48 vectors

$$\Phi = \{\pm\epsilon_i\}_{1 \leq i \leq 4} \cup \{\pm\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 4} \cup \left\{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\}.$$

We can think of  $\Phi$  as the set of all  $\alpha \in \bigoplus \mathbb{Q}\epsilon_i$  such that  $2\alpha \in \bigoplus \mathbb{Z}\epsilon_i$  and  $\alpha$  has integral square length  $\|\alpha\|^2 \leq 2$ . Also,  $\Pi := \{\epsilon_4, \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4), \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \}$  forms a set of simple roots (cf. [Bo, Appendix, Plate VIII]). Now take  $l = 3$  and let  $\lambda = 0$ . Since  $\rho = \frac{1}{2}(11\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + \epsilon_4)$ , it is directly checked that  $\Phi_0$  has

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 + \epsilon_4\} \cup \left\{-\epsilon_3, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)\right\}$$

as a set of simple roots. Thus,  $\Phi_0$  has type  $A_2 \times A_2$ , and there is clearly no  $J \subset \Pi$  for which  $\Phi_J$  has type  $A_2 \times A_2$ . More generally, still for type  $F_4$ , if 3 divides  $l$ , say  $l = 3l'$ , then the conclusion of the lemma fails for any  $\lambda = (l' - 1)\rho$ .

On the other hand,  $l = 9$  satisfies Assumption 1.2.1, but 9 is not good for  $F_4$ . Here  $\Phi_0 = \{\pm\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)\}$  has type  $A_1$ , so  $\Phi_0$  does satisfy the conclusion of the lemma. We will see in Theorem 3.4.1 below that this fact holds generally as long as  $l$  satisfies Assumption 1.2.1.

**3.1.1. Richardson orbits.** Let  $G$  be a complex, simple and simply connected algebraic group over  $\mathbb{C}$  with root system  $\Phi$ . For  $J \subset \Pi$ , the (standard) parabolic subgroup  $P_J = L_J \ltimes U_J \supseteq B$  of  $G$  has a dense (open) orbit  $\mathcal{C}'_J$  in the Lie algebra  $\mathfrak{u}_J$  of  $U_J$  under the adjoint action of  $P_J$ . In particular, if  $J = \emptyset$ , then  $L_J = T$ , and  $P_J = B$ , the Borel subgroup corresponding to  $\Phi^-$ . The corresponding Richardson orbit  $\mathcal{C}_J$  is the  $G$ -orbit  $G \cdot x$  for any  $x \in \mathcal{C}'_J$ . Therefore, when  $J = \emptyset$ ,  $\mathcal{C}_J$  is the regular or principal nilpotent orbit. Also, it is straightforward to show that the (Zariski) orbit closure  $\overline{\mathcal{C}}_J$  equals  $G \cdot \mathfrak{u}_J$ . In addition,  $\overline{\mathcal{C}}_J$  has dimension  $2 \dim \mathfrak{u}_J$ .

If  $J, K$  are  $W$ -conjugate subsets of  $\Pi$ , the Johnston-Richardson theorem states that  $\mathcal{C}_J = \mathcal{C}_K$ . Hence, given  $\lambda \in X$ , if there exists  $w \in W$  and  $J \subseteq \Pi$  with  $w(\Phi_\lambda) = \Phi_J$ , then  $\lambda$  defines in a unique way a Richardson class  $\mathcal{C}_\lambda$  in  $\mathcal{N}$  by setting  $\mathcal{C}_\lambda = \mathcal{C}_J$ . For  $\lambda \in X$  with  $w(\Phi_\lambda) = \Phi_J$  as above, set  $\mathcal{N}(\Phi_\lambda) := G \cdot \mathfrak{u}_J = \overline{\mathcal{C}}_J \subseteq \mathcal{N}$ .

For more details on the above results, see [Hum1, Chap. 5]. An important generalization of Richardson classes (namely, induced classes) will be discussed when it is needed in §3.5.

### 3.2. The case of the classical Lie algebras

In [CLNP, §3.1-3.7], explicit determinations were given for all irreducible root systems  $\Phi$  of a  $J \subset \Pi$  so that  $w(\Phi_0) = \Phi_J$  when  $l = p$  is a good prime. The proofs there work equally well when  $l$  satisfies Assumption 1.2.1. As noted before, we use the root notation and ordering of Bourbaki [Bo]. The results below will be useful in Chapter 4.

The first theorem below treats the cases when  $\Phi$  has type  $A$  or  $B$ , and the second theorem below summarizes the situation in types  $C$  and  $D$ . Since  $\dim \mathcal{N}(\Phi_0) = 2 \dim \mathfrak{u}_J$ , we have  $\dim \mathcal{N}(\Phi_0) = |\Phi| - |\Phi_0|$ . Also, in types  $A$ – $D$ , Assumption 1.2.1 means that  $l$  is good, so that Lemma 3.1.1 is applicable.<sup>1</sup>

**Theorem 3.2.1.** *Let  $l$  be as in Assumption 1.2.1,  $\mathfrak{g}$  be a classical simple Lie algebra with  $\Phi$  of type  $A_n$  (respectively,  $B_n$ ), and  $h = n + 1$  (respectively,  $2n$ ) be the Coxeter number of  $\Phi$ .*

- (a) *If  $l \geq h$  then  $\mathcal{N}(\Phi_0) = \mathcal{N}(\mathfrak{g})$  and  $\dim \mathcal{N}(\Phi_0) = |\Phi|$ .*
- (b) *Suppose that  $l < h$  where  $h - 1 = lm + s$  with  $m > 0$  and  $0 \leq s \leq l - 1$ . Then  $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$  where  $J \subseteq \Pi$  such that when*
  - (i)  *$\Phi$  is of type  $A_n$ ,*

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{l-s-1 \text{ times}};$$

*where  $\dim \mathcal{N}(\Phi_0) = n(n+1) - m(lm + 2s - l + 2)$ .*

- (ii)  *$\Phi$  is of type  $B_n$ ,*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-1}{2} \text{ times}} \times B_{\frac{m+1}{2}} & \text{if } s \text{ is even (} m \text{ odd),} \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s+1}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-2}{2} \text{ times}} \times B_{\frac{m}{2}} & \text{if } s \text{ is odd (} m \text{ even).} \end{cases}$$

*Also,*

$$\dim \mathcal{N}(\Phi_0) = \begin{cases} 2n^2 - \frac{m(lm+2s-l+3)+1}{2} & \text{if } s \text{ is even (} m \text{ odd),} \\ 2n^2 - \frac{m(lm+2s-l+3)}{2} & \text{if } s \text{ is odd (} m \text{ even).} \end{cases}$$

**Theorem 3.2.2.** *Let  $l$  be as in Assumption 1.2.1,  $\mathfrak{g}$  be a classical simple Lie algebra with  $\Phi$  of type  $C_n$  (respectively,  $D_n$ ), and  $h = 2n$  (respectively,  $2n - 2$ ) be the Coxeter number of  $\Phi$ .*

- (a) *If  $l \geq h$  then  $\mathcal{N}(\Phi_0) = \mathcal{N}(\mathfrak{g})$  and  $\dim \mathcal{N}(\Phi_0) = |\Phi|$ .*
- (b) *Suppose that  $l < h$  where  $h + 1 = lm + s$  with  $m > 0$  and  $0 \leq s \leq l - 1$ . Then  $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$  where  $J \subseteq \Pi$  such that when*
  - (i)  *$\Phi$  is of type  $C_n$ ,*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-1}{2} \text{ times}} \times C_{\frac{m-1}{2}} & \text{if } s \text{ is even,} \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s-1}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s}{2} \text{ times}} \times C_{\frac{m}{2}} & \text{if } s \text{ is odd.} \end{cases}$$

<sup>1</sup>In the statement of the theorem, root system terms  $A_0, B_0, \dots$  should be ignored.



Also,

$$\dim \mathcal{N}(\Phi_0) = \begin{cases} 2n^2 - \frac{m(lm+2s-l-1)+1}{2} & \text{if } s \text{ is even (} m \text{ odd),} \\ 2n^2 - \frac{m(lm+2s-l-1)}{2} & \text{if } s \text{ is odd (} m \text{ even).} \end{cases}$$

(iv)  $\Phi$  is of type  $D_n$ ,

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s-1}{2} \text{ times}} \times D_{\frac{m+1}{2}} & \text{if } s \text{ is even and } m \geq 3, \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s-1}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s}{2} \text{ times}} \times D_{\frac{m+2}{2}} & \text{if } s \text{ is odd,} \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2} \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{l-s+1}{2} \text{ times}} & \text{if } s \text{ is even and } m = 1. \end{cases}$$

Also,

$$\dim \mathcal{N}(\Phi_0) = \begin{cases} 2n^2 - 2n - \frac{m(lm+2s-l+1)-1}{2} & \text{if } s \text{ is even (} m \text{ odd),} \\ 2n^2 - 2n - \frac{m(lm+2s-l+1)}{2} & \text{if } s \text{ is odd (} m \text{ even).} \end{cases}$$

### 3.3. The case of the exceptional Lie algebras

For the exceptional types  $G_2, F_4, E_6, E_7$ , and  $E_8$ , the precise determination of the subroot system  $\Phi_0$  and the variety  $\mathcal{N}(\Phi_0)$  can be carried out by hand. In most cases, the determination of this variety can be deduced from its dimension ( $\dim \mathcal{N}(\Phi_0) = |\Phi| - |\Phi_0|$ ) and the fact that  $\mathcal{N}(\Phi_0)$  is the closure of a Richardson orbit; see [CLNP, §4.2]. When this information is not sufficient, the correct Richardson orbit can be pinned down by using the Weyl group as discussed in [CLNP, §4.3]. In fact, for computational purposes in Chapter 4, for each value of  $l$  satisfying Assumption 1.2.1, we identify an explicit element  $w \in W$  and a subset  $J \subset \Pi$  such that  $w(\Phi_0^+) = \Phi_J^+$ . As observed before, the choices of  $w$  and  $J$  are not unique in general. The computer package MAGMA [BC, BCP] was used to verify these facts. The tables providing the description of  $\mathcal{N}(\Phi_0)$ ,  $w$ , and  $J$  for various possible values of  $l$  are presented in the Appendix A.1.

### 3.4. Standardizing $\Phi_0$

If  $l$  satisfies Assumption 1.2.1 and if  $\Phi$  has classical type, then  $l$  is automatically good for  $\Phi$ . Thus, in this case, the theorem below follows immediately from Lemma 3.1.1. However, in the exceptional types, we need to quote the computer results tabulated in Appendix A.1 in case  $l$  is not good for  $\Phi$  (but still satisfies Assumption 1.2.1). We essentially worked out the case of  $F_4$  after the statement of Lemma 3.1.1, where  $l = 9$  is the only value that needs to be considered.

For future reference, we summarize this result as follows.

**Theorem 3.4.1.** *Let  $l$  be as in Assumption 1.2.1. Then there exists  $w \in W$  and a subset  $J \subseteq \Pi$  such that  $w(\Phi_{0,l}^+) = \Phi_J^+$ .*

### 3.5. Normality of orbit closures

We consider certain nilpotent orbit closures for a complex simple Lie algebra  $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$  having root system  $\Phi$ , etc. Let  $G$  be a complex (connected) algebraic group of the same root type as  $\mathfrak{g}$ . Since we are only interested in the adjoint action of  $G$  on its Lie algebra  $\mathfrak{g}$ , we do not require  $G$  to be simply connected. In case  $\Phi$  has type  $B_n$  (resp.,  $C_n, D_n$ ), we will take  $G = SO_{2n+1}(\mathbb{C})$  (resp.,  $Sp_{2n}(\mathbb{C}), SO_{2n}(\mathbb{C})$ ).

In the discussion below we will make use of the considerable work available in determining normal  $G$ -orbit closures in the nilpotent variety  $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ .

Fix an integer  $l$  satisfying Assumption 1.2.1, and let  $\Phi_0 = \Phi_{0,l}$ . We are especially interested when the variety  $\mathcal{N}(\Phi_0)$  is normal. In fact, we will verify that, in almost all cases, it is normal.

**3.5.1. The classical case.** When  $\Phi$  has type  $A_n$ , a famous result of Kraft-Procesi [KP1, §0, Theorem] states all orbit closures in  $\mathcal{N}$  are normal varieties. In this case, nilpotent orbits in  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$  are naturally in one-to-one correspondence with the set  $\mathcal{P}(n+1)$  of partitions  $\eta = (\eta_1, \eta_2, \dots, \eta_{n+1})$  of  $n+1$ . Here  $\eta_1 \geq \eta_2 \geq \dots \geq \eta_n \geq 0$  and  $\eta_1 + \dots + \eta_{n+1} = n+1$ . We also write  $\eta \vdash n+1$  to mean that  $\eta \in \mathcal{P}(n+1)$ . Also, if a part  $a$  is repeated  $b$  times, we often write  $a^b$  in place of  $\underbrace{a, \dots, a}_b$

in  $\eta$ . Thus, if  $\mathcal{O}_\eta$  denotes the corresponding orbit, the elements  $x \in \mathcal{O}_\eta$  are just those nilpotent  $(n+1) \times (n+1)$ -matrices which have Jordan blocks of sizes  $\eta_i \times \eta_i$ ,  $1 \leq i \leq n$ . A parabolic subgroup  $P_J$  also determines a *composition*  $\eta = \eta_J = (\eta_1, \dots, \eta_{n+1})$  of  $n+1$  meaning that each  $\eta_i \geq 0$  and  $\eta_1 + \dots + \eta_{n+1} = n+1$ . Thus, when displayed in the usual way as matrices, the Levi factor  $L_J$  of  $P_J$  corresponds to blocks of sizes  $\eta_1 \times \eta_1, \dots, \eta_{n+1} \times \eta_{n+1}$  down the diagonal. Two Levi factors  $L_J$  and  $L_K$  are conjugate in the group  $G = SL_{n+1}(\mathbb{C})$  if and only if the partitions defined by rearrangement of the two compositions  $\eta_J$  and  $\eta_K$  are equal. We let  $\tilde{\eta}_J$  denote this partition. A well-known result of Kraft states that  $\mathcal{N}(\Phi_J)$  is the closure in  $\mathcal{N}$  of the nilpotent class defined by the partition  $\tilde{\eta}'_J$  dual to  $\tilde{\eta}_J$ . In this way, the Richardson orbits in type  $A_n$  described in Theorem 3.2.1 can be explicitly identified as certain  $\mathcal{O}_\eta$ ,  $\eta \vdash n+1$ .

The set  $\mathcal{P}(n)$  (as well as subsets considered below) is partially ordered by putting  $\eta \leq \sigma$  provided that, for each  $i$ ,  $\eta_1 + \dots + \eta_i \leq \sigma_1 + \dots + \sigma_i$ . Then, given  $\eta, \sigma \in \mathcal{P}(n+1)$ ,  $\mathcal{O}_\eta \subseteq \overline{\mathcal{O}_\sigma}$  if and only if  $\eta \leq \sigma$ . For a nonempty subset  $\Gamma \subseteq \mathcal{P}(n)$ , a pair in  $\Gamma$  is a pair  $(\eta, \sigma)$  of distinct elements in  $\Gamma$  such that  $\eta \leq \sigma$ . If there is no other element  $\tau \in \Gamma$  which is strictly between  $\eta$  and  $\sigma$  in the ordering  $\leq$ , the pair is called minimal.

For the classical types  $B_n$  and  $C_n$ , the nilpotent classes are also in natural one-to-one correspondence with certain sets of partitions. Thus, for a positive integer  $N$ , let  $\mathcal{P}_1(N)$  be the set of partitions  $\eta \vdash N$  in which each even part  $\eta_i$  is repeated an even number of times. Similarly, if  $N$  is even, let  $\mathcal{P}_{-1}(N)$  consist of those  $\eta \vdash N$  in which each odd part  $\eta_i$  is repeated an even number of times. When  $\Phi$  has type  $B_n$  (resp.,  $C_n$ ), the  $G$ -orbits on  $\mathcal{N}$  correspond naturally to the  $\eta \in \mathcal{P}_1(2n+1)$  (resp.,  $\eta \in \mathcal{P}_{-1}(2n)$ ). In fact, given  $\eta \in \mathcal{P}_\epsilon(N)$  ( $\epsilon = \pm 1$ ), the corresponding orbit is  $\mathcal{O}_{\epsilon, \eta} := \mathcal{O}_\eta \cap \mathfrak{g}$ , i.e., it is the intersection of the nullcone  $\mathcal{N}$  of  $G$  with the corresponding  $SL_{2n+1}(\mathbb{C})$ -orbit (resp.,  $SL_{2n}(\mathbb{C})$ -orbit) indexed by  $\eta$ .

If  $\Phi$  has type  $D_{2n}$ , then the  $G$ -orbits in  $\mathcal{N}$  correspond to the elements in  $\mathcal{P}_1(2n)$ , except in the case in which all the parts of  $\eta$  are even (i.e.,  $\eta$  is very even); in this case,  $\eta$  defines two orbits  $\mathcal{O}_\eta^I$  and  $\mathcal{O}_\eta^{II}$  (which combine under the action of the full orthogonal group  $O(2n)$  on  $\mathfrak{g}$ ).

If  $\eta \in \mathcal{P}_\epsilon(m)$  for some positive integer  $m$ , call  $\eta$  an  $\epsilon$ -partition. Given a pair  $(\eta, \sigma)$  in  $\mathcal{P}_\epsilon(n)$ , suppose that  $\eta_i = \sigma_i$ ,  $1 \leq i \leq r$ , and  $\eta'_j = \sigma'_j$ ,  $1 \leq j \leq s$ . In other words, the first  $r$  rows and first  $s$  columns in the Young diagrams associated to  $\eta$  and  $\sigma$  are the same. Suppose that  $(\eta_1, \dots, \eta_r)$  is an  $\epsilon$ -partition, and consider the pair  $(\bar{\eta}, \bar{\sigma})$  in  $\mathcal{P}_{(-1)^s \epsilon}(m)$  obtained by letting  $\bar{\eta} = (\eta_{r+1}, \dots)$ ,  $\bar{\sigma} = (\sigma_{s+1}, \dots)$ , and  $m = \eta_{r+1} + \dots + \eta_n$ . In this case, write  $(\eta, \sigma) \rightarrow (\bar{\eta}, \bar{\sigma})$ , allowing the possibility that no rows or columns were removed, i.e.,  $\bar{\sigma} = \sigma$  and  $\bar{\eta} = \eta$ . Then a pair  $(\eta, \sigma)$  is called irreducible if  $(\eta, \sigma) \rightarrow (\bar{\eta}, \bar{\sigma})$  implies that  $(\eta, \sigma) = (\bar{\eta}, \bar{\sigma})$ . By means of this process, every irreducible pair  $(\eta, \sigma)$  in  $\mathcal{P}_\epsilon(n)$  can be reduced to an irreducible minimal pair  $(\bar{\eta}, \bar{\sigma})$  in  $\mathcal{P}_{\epsilon'}(m)$ . The irreducible minimal pairs  $(\eta, \sigma)$  are classified in [KP2, 3.4]; there are precisely 8 distinct minimal irreducible pairs. Of course, a pair  $(\eta, \sigma)$  may be minimal irreducible in say  $\mathcal{P}_1(2n)$ , but not minimal irreducible when regarded as a pair in  $\mathcal{P}_{-1}(2n)$ .

A main result, stated in [KP2, Thm. 16.2], establishes that, given  $\sigma \in \mathcal{P}_\epsilon(n)$ , the orbit closure  $\overline{\mathcal{O}_{\epsilon,\sigma}}$  is normal in a codimension 2 class  $\mathcal{O}_{\epsilon,\eta} \subset \overline{\mathcal{O}_{\epsilon,\sigma}}$  if and only if the irreducible minimal pair  $(\overline{\eta}, \overline{\sigma}) \in \mathcal{P}_{\epsilon'}(n')$  obtained from  $(\eta, \sigma)$  (by the row and column removal process described above) is not the pair

$$(3.5.1) \quad (2m, 2m), (2m-1, 2m-1, 1, 1) \in \mathcal{P}_1(4m) \times \mathcal{P}_1(4m).$$

So  $n' = 4m$  in this case. We will use this result to study the normality of the  $\mathcal{N}(\Phi_0)$ .

Given  $\eta \in \mathcal{P}(2n+1)$ , there exists a unique  $\eta \in \mathcal{P}_1(2n+1)$  which is largest partition (with respect to  $\trianglelefteq$ ) among all  $\sigma \in \mathcal{P}_1(2n+1)$  satisfying  $\sigma \trianglelefteq \eta$ . It is denoted  $\eta_B$  and it is called the  $B$ -collapse of  $\eta$ . See [CM, Lemma 6.3.3] where its (simple) construction is indicated. Similarly, if  $\eta \in \mathcal{P}_1(2n)$  (resp.,  $\eta \in \mathcal{P}_{-1}(2n)$ ), there is a unique largest  $\eta_D$  (resp.,  $\eta_C$ ) in  $\mathcal{P}_1(2n)$  (resp.,  $\mathcal{P}_{-1}(2n)$ ) among those elements  $\sigma \in \mathcal{P}_1(2n)$  (resp.,  $\sigma \in \mathcal{P}_{-1}(2n)$ ) satisfying  $\sigma \trianglelefteq \eta$ .

The extension of Kraft's result described above to the other classical types requires the notion of an induced nilpotent class. Thus, let  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$  be a parabolic subalgebra of  $\mathfrak{g}$ . For any nilpotent class  $\mathcal{O}_\mathfrak{l}$  in  $\mathfrak{l}$ , the parabolic subgroup  $P$  with Lie algebra  $\mathfrak{p}$  has a unique open (nilpotent) orbit  $\mathcal{O}'_\mathfrak{l}$  in  $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ , which in turn defines a nilpotent orbit, denoted  $\text{ind}_\mathfrak{l}^\mathfrak{g} \mathcal{O}_\mathfrak{l}$  since it can be proved that it does not depend on the choice of parabolic subalgebra  $\mathfrak{p}$  having Levi factor  $\mathfrak{l}$ . Furthermore, induction of nilpotent classes is transitive, i.e., for  $\mathfrak{l}_1 \subseteq \mathfrak{l}_2 \subseteq \mathfrak{l}_3$ ,  $\text{ind}_{\mathfrak{l}_1}^{\mathfrak{l}_3} = \text{ind}_{\mathfrak{l}_2}^{\mathfrak{l}_3} \circ \text{ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}$  on nilpotent orbits in  $\mathfrak{l}_1$ . In addition, if  $\mathcal{O}_J$  is the Richardson class defined by  $J \subset \Pi$ , then  $\mathcal{O}_J = \text{ind}_{\mathfrak{l}_J}^\mathfrak{g} \mathcal{O}_0$ , where  $\mathcal{O}_0$  denotes the trivial class in  $\mathfrak{l}_J$ . Since  $[\mathfrak{l}_J, \mathfrak{l}_J]$  is a direct sum of classical simple Lie algebras,  $\mathcal{O}_0$  is defined by a single column partition  $(1^s)$  on each simple component. Therefore, using [CM, Thm. 7.3.3],  $\mathcal{O}_J$  can be explicitly described as a nilpotent class  $\mathcal{O}_{\epsilon,\eta}$ .

Write  $N = m'l + s'$  where  $0 \leq s' \leq l-1$ . Then  $\mathcal{N}(\Phi_0) = \overline{\mathcal{O}_{\sigma_X}}$  where  $\sigma = (l^{m'}, s')$  and  $\sigma_X$  is the  $X$ -collapse of  $\sigma$  ( $X \in \{B, C, D\}$ ). See [UGA1] for more details. Now we analyze each type.

Case 1:  $\Phi$  has type  $B_n$ . First, suppose that  $s'$  is odd. Then  $\sigma_B = (l^{m'}, s') = \sigma$ . In this case, because  $l$  is odd, it is impossible to reduce  $\sigma$  to a partition  $(2m, 2m) \in \mathcal{P}_1(4m)$  (which occurs in (3.5.1)) by removing rows  $\sigma_1, \dots, \sigma_r$  (which automatically form an  $\epsilon$ -partition) and an *even* number of columns  $(\sigma'_1, \dots, \sigma'_s)$ . Therefore,  $\mathcal{N}(\Phi_0)$  is normal in this case.

Case 2:  $\Phi$  has type  $B_n$  and  $s'$  is even. Then  $\sigma_B = \sigma$  if  $s' = 0$  and  $\sigma_B = (l^{m'}, s'-1, 1)$  if  $s' > 0$ . Again, it is clearly impossible to reduce such a partition to one of the form  $(2m, 2m)$  by removing rows and an even number of columns. Therefore,  $\mathcal{N}(\Phi_0)$  is normal in this case.

Case 3:  $\Phi$  has type  $D_n$ , thus  $2n = m'l + s'$ . If  $s'$  is odd or 0, then  $\sigma_D = \sigma$ , and the situation is very similar to that in Case 1, and normality follows. If  $s'$  is a positive even integer, then  $\sigma_D = (l^{m'}, s'-1, 1)$ , placing in the same situation as in Case 2. Thus,  $\mathcal{N}(\Phi)$  is normal in type  $D$ .

Case 4:  $\Phi$  has type  $C_n$ ,  $2n = m'l + s'$ , and  $s'$  is even (thus  $m'$  is even). Then  $\sigma_C = \sigma = (l^{m'}, s') \in \mathcal{P}_{-1}(2n)$ . Now we must remove an *odd* number  $t$  of columns to get to some  $\mathcal{P}_1(4m)$ . Clearly, in order to obtain  $(2m, 2m)$ , we must have  $s' < t < l$ . Thus, we can assume that  $s' + 1 < l$ . The possible  $\eta$  for which  $(\eta, \sigma)$  is minimal in  $\mathcal{P}_{-1}(2n)$  are

$$\begin{cases} \eta_1 = (l^{m'-2}, l-1, l-1, s'+2); \\ \eta_2 = (l^{m'}, 1, 1), \quad s' = 2; \\ \eta_3 = (l^{m'}, s'-2, 2), \quad s' > 2. \end{cases}$$

But because  $s' < t$ , it is impossible to obtain  $(2m-1, 2m-1, 1, 1)$  from any of these partitions while obtaining  $(2m, 2m)$  from  $\sigma$  by removing rows and (an even number  $t > s'$ ) of columns.

Case 5:  $\Phi$  has type  $C_n$ ,  $2n = m'l + s'$ , and  $s'$  is odd (thus  $m'$  is odd). In this case,

$$\sigma_C = (l^{m'-1}, l, s' + 1).$$

If  $l - 1 > s' + 1$ , it is impossible to obtain a partition  $(2a, 2a)$ ,  $a > 0$ , from  $\sigma_C$  by removing rows (from top to bottom) and columns (from left to right). If  $l - 1 = s' + 1$ , then all the  $m' - 1$  rows of length  $l$  must be removed and then an *odd* number of columns must be removed. Since  $l - 1 = s' + 1$  is even, it is again not possible to arrive at a partition  $(2a, 2a)$ ,  $a > 0$ .

We conclude in all classical cases that  $\mathcal{N}(\Phi_0)$  is normal.

**3.5.2. Exceptional cases.** In case  $J \subseteq \Pi$  consists of mutually orthogonal short roots, an important result of Broer [Br2, Thm. 4.1] can be applied, establishing that  $\mathcal{N}(\Phi_J)$  is a normal variety. In particular, this applies to the full nullcone  $\mathcal{N} = \mathcal{N}(\emptyset)$  (a well-known result of Kostant) and the closure  $\mathcal{N}_{\text{subreg}} = \overline{\mathcal{N}(\{\pm\alpha\})}$ ,  $\alpha \in \Pi$  (short), of the subregular class. Using the tables in Appendix A.1 (where  $J$  is explicitly described), we can treat the various exceptional types below. Notice that  $\mathcal{N}_{\text{subreg}}$  is the closure of the unique nilpotent class of codimension 2 in  $\mathcal{N}$ .

Type  $G_2$ . The relevant  $\mathcal{N}(\Phi_0)$  are either the full nullcone  $\mathcal{N}$  (for  $l \geq 6$ ) or  $\mathcal{N}_{\text{subreg}}$ . As remarked above, these are normal.

Type  $F_4$ . There are four possible orbits, having Carter-Bala labels  $F_4(a_2)$ ,  $F_4(a_2)$ ,  $F_4(a_1)$ , and  $F_4$  and corresponding distinguished Dynkin diagrams labeled (0200), (0202), (2202), and (2222), respectively (using [CM, p. 128]). By [Br1, Thm. 1], the corresponding orbit closures are normal.

Type  $E_6$ . There are four relevant orbits having Bala-Carter labels  $A_4 + A_1$ ,  $E_6(a_3)$ ,  $E_6(a_1)$ , and  $E_6$ . The last three are Richardson classes, with Levi factor root systems  $A_1 \times A_1 \times A_1$ ,  $A_1$ , and  $\emptyset$ . Thus, Broer's theorem [Br2, Thm. 4.1] applies to guarantee these have normal orbit closures. The final case  $A_4 + A_1$  has normal orbit closure, by [So1, p. 296].

Type  $E_7$ . There are seven relevant orbits having Bala-Carter labels  $A_4 + A_2$ ,  $A_6$ ,  $E_6(a_1)$ ,  $E_7(a_1)$ ,  $E_7(a_2)$ ,  $E_7(a_3)$ , and  $E_7$ . Again, [Br2, Thm. 4.1] implies the last five have normal orbit closures. Using techniques from [So1] and [So2], Sommers [So3] has informed us that he has verified the normality for the remaining two cases (unpublished).

Type  $E_8$ . There are nine relevant orbits having Bala-Carter labels  $A_6 + A_1$ ,  $E_8(b_6)$ ,  $E_8(a_6)$ ,  $E_8(a_5)$ ,  $E_8(a_4)$ ,  $E_8(a_3)$ ,  $E_8(a_2)$ ,  $E_8(a_1)$ , and  $E_8$ . The last five have normal orbit closures, again by [Br2, Thm. 4.1]. Again, Sommers has informed us that he has verified normality for  $E_8(a_6)$  and  $E_8(a_5)$ . The remaining two cases,  $A_6 + A_1$  (when  $l = 7$ ) and  $E_8(b_6)$  (when  $l = 9$ ) remain open at present.

**3.5.3. Summary.** We summarize the analysis in the following theorem.

**Theorem 3.5.1.** *Let  $l$  be as in Assumption 1.2.1 and  $J \subseteq \Pi$  so that  $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$ . If  $\Phi$  is of type  $E_8$ , assume that  $l \neq 7, 9$ . Then  $\mathcal{N}(\Phi_0)$  is a normal variety.*

### 3.6. Resolution of singularities

We maintain the notation of the previous section. Let  $J \subseteq \Pi$  and  $P_J$  be the associated parabolic subgroup. From the Bruhat decomposition, it follows that the quotient map  $G \xrightarrow{\pi} G/P_J$  has local sections in the sense that  $G/P_J$  has an open covering by affine spaces  $X \cong \mathbb{A}^{\dim \mathfrak{u}_J}$  such that  $\pi^{-1}X \cong X \times P_J$ . Thus, the orbit space  $G \times^{P_J} \mathfrak{u}_J := (G \times \mathfrak{u}_J)/P_J$  for the natural right action of  $P_J$  on  $G \times \mathfrak{u}_J$  satisfies

$$\mathbb{C}[G \times^{P_J} \mathfrak{u}_J] \cong (\mathbb{C}[G] \otimes S^\bullet(\mathfrak{u}_J^*))^{P_J} =: \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*).$$

Also, the natural projection  $G \times^{P_J} \mathfrak{u}_J \rightarrow G/P_J$  identifies  $G \times^{P_J} \mathfrak{u}_J$  with the cotangent bundle of the smooth variety  $G/P_J$ . For background and more discussion of the topics in this section, see [Jan3, pp. 90–97].

In addition, there is the moment (or collapsing) map  $\mu : G \times^{P_J} \mathfrak{u}_J \rightarrow G \cdot \mathfrak{u}_J$  defined by mapping the  $P_J$ -orbit  $[x, u]$  of  $(x, u) \in G \times \mathfrak{u}_J$  to  $x \cdot u \in G \cdot \mathfrak{u}_J$ . Then  $\mu$  is a desingularization of  $G \cdot \mathfrak{u}_J$ , in the sense that it is a birational, proper morphism of varieties, if and only if the following condition holds:

$$(3.6.1) \quad \text{for all } x \in \mathcal{C}'_J, C_G(x) = C_{P_J}(x).$$

Recall that the condition  $x \in \mathcal{C}'_J$  means just that  $P_J \cdot x$  is dense in  $\mathfrak{u}_J$ . It is well known that  $C_G(x)^o = C_{P_J}(x)^o$ , i. e., that the two centralizers  $C_G(x)$  and  $C_{P_J}(x)$  have the same connected components of the identity; see [Car, Cor. 5.2.2].

**Lemma 3.6.1.** (a) *If  $\mu$  is a desingularization, then*

$$\mathbb{C}[G \times^{P_J} \mathfrak{u}_J] \cong \mathbb{C}[\mu^{-1}\mathcal{C}_J] \cong \mathbb{C}[\mathcal{C}_J],$$

*where, for a complex variety  $X$ ,  $\mathbb{C}[X]$  denotes the algebra of regular functions on  $X$ . (Recall that  $\mathcal{C}_J := G \cdot x$  for any  $x \in \mathcal{C}'_J$ .)*

(b) *If  $x$  is an even nilpotent element (in the sense of Bala-Carter, see [CM, Ch. 8]), condition (3.6.1) on centralizers holds.*

PROOF. (a) holds by [Jan3, Remark, p. 95], and (b) follows from [Jan3, Remark, p. 93].  $\square$

In the theorem below, we show that (3.6.1) holds in all the situations of interest in this paper.

If  $\mu$  is a desingularization, and if, in addition,  $\overline{\mathcal{O}}_J = G \cdot \mathfrak{u}_J$  is a normal variety, then

$$(3.6.2) \quad \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \times^{P_J} \mathfrak{u}_J] \cong \mathbb{C}[G \cdot \mathfrak{u}_J].$$

This follows because  $\overline{\mathcal{O}}_J \setminus \mathcal{O}_J$  has codimension at least 2.

Now we can state the following result.

**Theorem 3.6.2.** *Let  $l$  be as in Assumption 1.2.1 and choose  $J \subseteq \Pi$  so that  $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$ .*

(a) *The moment map  $\mu : G \times^{P_J} \mathfrak{u}_J \rightarrow G \cdot \mathfrak{u}_J$  is a  $G$ -equivariant desingularization of  $G \cdot \mathfrak{u}_J$ .*

(b) *If  $\Phi$  is of type  $E_8$ , assume that  $l \neq 7, 9$ . Then the identifications (3.6.2) hold.*

PROOF. We first prove that the condition (3.6.1) holds, namely, that  $C_G(x) \subseteq P_J$ , where  $x \in \mathfrak{u}_J$ . Then (a) holds by the discussion above Lemma 3.6.1. Consequently, Theorem 3.5.1 and the discussion right above establishes (b).

The verification of (3.6.1) will be case-by-case.

Case 1:  $\Phi$  has type  $A_n$ . Without loss of generality we can assume that  $G = GL_n(k)$ . The centralizer is connected so  $C_G(x) = C_G(x)^0$ . Since  $x$  is Richardson we have by [Car, Corollary 5.2.4]  $C_G(x)^0 \subseteq P_J$ .

Case 2:  $\Phi$  has type  $B_n$ . Let  $N = 2n + 1$  and, as before, write  $N = lm' + s'$  where  $0 \leq s' \leq l - 1$  and  $m' > 0$ . Set  $\eta = (l^{m'}, s')$  and recall that  $\mathcal{N}(\Phi_0) = \overline{\mathcal{O}}_{\eta_B}$  where  $\eta_B$  is the  $B$ -collapse of  $\eta$ .

For type  $B_n$  we have

$$\eta_B = \begin{cases} (l^{m'}, s') & \text{if } s' \text{ is odd or } s' = 0, \\ (l^{m'}, s' - 1, 1) & \text{if } s' \text{ is even and } s' \neq 0. \end{cases}$$

In either case each of the nonzero parts are odd so the associated weighted Dynkin diagram has even entries [CM, §5.3]. Therefore, the orbit  $\mathcal{O}_{\eta_B}$  is even, and the centralizer  $C_G(x)$  is contained in  $P_J$  by Lemma 3.6.1.

Case 3:  $\Phi$  has type  $D_n$ . This case is similar to that in Case 2, and it is left to the reader.

Case 4:  $\Phi$  has type  $C_n$ . Now let  $N = 2n$  and write  $N = lm' + s'$ . Then

$$\eta_C = \begin{cases} (l^{m'}, s') & \text{if } m' \text{ is even (} s' \text{ even),} \\ (l^{m'-1}, l-1, s'+1) & \text{if } m' \text{ is odd (} s' \text{ odd).} \end{cases}$$

In the first case, when  $m'$  and  $s'$  are both even, let  $b$  be the number of distinct even nonzero parts. So  $b = 1$  and by [CM, p. 92] the component group  $A(\mathcal{C}_{\eta_C}) \cong (\mathbb{Z}/2\mathbb{Z})^{b-1}$  is trivial. Here the component group is defined as  $C_{G'}(x)/C_{G'}(x)^o$  for any  $x \in \mathcal{C}'_J$ , where  $G'$  is the adjoint group  $PSp_{2n}$ . Thus,  $C_G(x)$  is generated by  $C_G(x)^o$  and the center of  $G$ , which is contained in  $P_J$ . Thus,  $C_G(x) \subseteq C_{P_J}(x)$  here also.

Now suppose that  $m'$  and  $s'$  are odd. In this case, the component group  $A(\mathcal{C}_J)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ , so a different line of reasoning is required. In [H, Cor. 7.7], Hesselink provides a necessary and sufficient criterion for having  $C_G(x) \subseteq P_J$  for  $x \in \mathcal{C}'_J$ . Here  $\eta_C$  satisfies condition (i) of [H, Cor. 7.7] where  $\epsilon = 1$  and  $\eta_C \in \text{Pai}(2n, m' - 1)$  (see [H, §6.1] for the definitions/notation).

Case 5:  $\Phi$  has exceptional type  $G_2, F_4, E_6, E_7$ , or  $E_8$ . In these cases, we refer to Appendix A.1, where the classes  $\mathcal{C}_J$  are all identified in the Bala-Carter notation. Using the tables given in [CM, pp. 128-134], we see that all but two of the orbits are even orbits (so that Lemma 3.6.1 is applicable). In the two other cases, namely, type  $A_4 + A_1$  in type  $E_6$  or type  $A_6 + A_1$  in type  $E_8$ , the full component group  $C_G(x)/C_G(x)^o$  is determined to be trivial there, where the component group is identified with the fundamental group  $\pi_1(\mathcal{C}_J)$  of the orbit. More directly, we can use the determination of this component group given in [CM] in all cases.

The theorem is completely proved. □



## CHAPTER 4

# Combinatorics and the Steinberg Module

In the computation of the cohomology algebra  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  for  $l > h$  in [GK], a key step is showing that the space of  $u_\zeta(\mathfrak{h})$ -invariants on  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$  is one dimensional (where  $\mathfrak{h} \subset \mathfrak{g}$  is the Cartan subalgebra). This fact follows because the space of  $u_\zeta(\mathfrak{h})$ -invariants on  $\Lambda_{\zeta, \emptyset}^\bullet$  is one dimensional. This fact is far from being true for  $l \leq h$ . For small  $l$ , a more intricate analysis is needed, namely, we must consider the multiplicity of a certain “Steinberg module” in  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ . This computation will then be used in Chapters 5 and 6 in order to complete the desired cohomology computations.

### 4.1. Steinberg weights

If  $l$  satisfies Assumption 1.2.1, by Theorem 3.4.1, we can choose  $w \in W$  and  $J \subseteq \Pi$  such that  $w(\Phi_0) = \Phi_J$ . Clearly, it can be additionally assumed that  $w(\Phi_0^+) = \Phi_J^+$ . In the classical cases, the particular choice of  $w$  and  $J \subseteq \Pi$  will not generally matter for the arguments that follow. However, when  $\Phi$  has type  $A_n$  with  $l$  dividing  $n + 1$ , a special  $w$  and  $J$  are identified in (4.8.1). Also, in the exceptional cases, each pair  $w \in W$  and  $J \subseteq \Pi$  identified in Appendix A.1 satisfies the property  $w(\Phi_0^+) = \Phi_J^+$ .

**Lemma 4.1.1.** *Let  $w \in W$  be such that  $w(\Phi_0^+) = \Phi_J^+$  for some  $J \subset \Pi$ . For all  $\alpha \in J$ ,  $\langle w \cdot 0, \alpha^\vee \rangle = l - 1$ .*

PROOF. Since  $w \cdot 0 = w(\rho) - \rho$ , the claim is equivalent to showing that  $l = \langle w(\rho), \alpha^\vee \rangle = \langle \rho, w^{-1}(\alpha)^\vee \rangle$  for all  $\alpha \in J$ . But, by our assumption on  $w$ ,  $w^{-1}(J)$  is the unique set  $\Pi_0$  of simple roots for  $\Phi_0$  contained in  $\Phi_0^+ = \Phi^+ \cap \Phi_0$ . So, the lemma asserts that  $\langle \rho, \beta^\vee \rangle = l$  for all  $\beta \in \Pi_0$ . Therefore, although  $w \in W$  is not uniquely determined, if the lemma holds for one choice of  $w$ , it holds for all choices. Now, in the exceptional cases, for each  $l$ , an element  $w \in W$  and a subset  $J \subseteq \Pi$  satisfying  $w(\Phi_0^+) = \Phi_J^+$  are identified in Appendix A.1. In these cases, the lemma can be checked directly by using the tables in Appendix A.2 which explicitly give  $w \cdot 0$ .

Now assume that  $\Phi$  has classical type  $A_n$ ,  $B_n$ ,  $C_n$ , or  $D_n$ . We can also assume that  $\Pi_0 \neq \emptyset$ . Then  $\Phi_0$  consists of all roots  $\alpha$  such that the coroot  $\alpha^\vee$  has height  $\text{ht}(\alpha^\vee) = \langle \rho, \alpha^\vee \rangle$  divisible by  $l$ . In particular,  $l < h$ . If  $\beta^\vee$  is any coroot of height  $m > 0$ , then, for any positive integer  $i$ ,  $3 \leq i < m$ , it is easy to see (in each possible case), that  $\beta^\vee = \delta^\vee + \gamma^\vee$  for  $\delta, \gamma \in \Phi$  satisfying  $\text{ht}(\delta^\vee) = i$ . Then  $\beta = a\delta + b\gamma$ , where  $a, b$  are positive rational numbers. If  $\beta \in \Phi_0^+$  with  $\text{ht}(\beta^\vee) = tl$ ,  $t > 1$ , then  $\beta^\vee = \delta^\vee + \gamma^\vee$  with  $\delta, \gamma \in \Phi_0^+$ . If  $\beta \in \Pi_0$ , it follows that  $\text{ht}(\beta^\vee) = l$ , i. e.,  $\langle \rho, \beta^\vee \rangle = l$ , as required.  $\square$

For chosen  $w \in W$  and  $J \subseteq \Pi$  such that  $w(\Phi_0^+) = \Phi_J^+$ , set

$$(4.1.1) \quad M := (\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*.$$

By the lemma,  $w \cdot 0$  is a  $J$ -Steinberg weight (see Section 2.10). The module  $M$  is therefore isomorphic to a “Steinberg” type module on  $U_\zeta(\mathfrak{l}_J)$  that remains irreducible if viewed as a  $u_\zeta(\mathfrak{l}_J)$ -module. The highest weight of  $M$  is  $-w_{0,J}(w \cdot 0)$  and the lowest weight of  $M$  is  $-w \cdot 0$ . Note that the module  $M$  does depend on the choice of  $w$ .



#### 4.2. Weights of $\Lambda_{\zeta, J}^\bullet$

By Section 2.7, the  $\overline{\text{Ad}}$ -action induces an action of  $U_\zeta(\mathfrak{p}_J)$  (and hence also of  $u_\zeta(\mathfrak{p}_J)$ ) on  $\mathcal{U}_\zeta(\mathfrak{u}_J)$ . This defines an action of  $U_\zeta(\mathfrak{p}_J)$  on the cohomology  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ . See also Section 2.8. In Theorem 4.3.1 below, we determine

$$\text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

By Proposition 2.9.1(b), it follows that, as a  $U_\zeta^0$ -module,  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$  is a subquotient of  $\Lambda_{\zeta, J}^\bullet$ . The key ingredients to proving Theorem 4.3.1 below are the following computational results concerning the weights in  $\Lambda_{\zeta, J}^\bullet$ .

**Proposition 4.2.1.** *Let  $l$  be as in Assumption 1.2.1. Choose  $w \in W$  and  $J \subseteq \Pi$  such that  $w(\Phi_0^+) = \Phi_J^+$ . Let  $\gamma$  be a  $J$ -dominant weight of  $\Lambda_{\zeta, J}^i$  such that  $\gamma = -w_{0, J}(w \cdot 0) + l\nu$  for some  $\nu \in X$ .*

- (a) *Suppose that  $l \nmid n+1$  when  $\Phi$  is of type  $A_n$  and  $l \neq 9$  when  $\Phi$  is of type  $E_6$ . Then  $\gamma = -w_{0, J}(w \cdot 0)$  (i.e.,  $\nu = 0$ ) and  $i = \ell(w)$ .*
- (b) *If  $\Phi$  is of type  $A_n$  with  $n+1 = l(m+1)$  and  $w$  is as defined in (4.8.1), then  $\gamma$  is one of the following, for  $0 \leq t \leq l-1$ :*

$$\gamma = -w_{0, J}(w \cdot 0) + l\varpi_{t(m+1)} \quad \text{with} \quad i = \ell(w) + (m+1)t(l-t).$$

*We set  $\varpi_0 = 0$ .*

- (c) *If  $\Phi$  is of type  $E_6$  and  $l = 9$  (assuming that  $w$  and  $J$  are as in Appendix A.1), then  $\gamma$  is one of the following:*

$$\begin{aligned} \gamma &= -w_{0, J}(w \cdot 0) \text{ with } i = \ell(w) = 8, \\ \gamma &= -w_{0, J}(w \cdot 0) + l\varpi_1 \text{ with } i = 20, \\ \gamma &= -w_{0, J}(w \cdot 0) + l\varpi_6 \text{ with } i = 20. \end{aligned}$$

One should observe that the weight  $\nu$  in the statement of the proposition must necessarily be  $J$ -dominant by Lemma 4.1.1. The proposition will be proved below. See Section 4.4.

#### 4.3. Multiplicity of the Steinberg module

Assuming that Proposition 4.2.1 holds, we can now determine how often the “Steinberg module”  $M$  (introduced in 4.1.1) appears in  $H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ . When  $l \geq h$ ,  $w = 1$ ,  $J = \emptyset$ , and  $M = \mathbb{C}$ . In parts (b) and (c) of the theorem, the notation  $l\varpi_j$  is used to denote the one-dimensional  $U_\zeta(\mathfrak{l}_J)$ -module with weight  $l\varpi_j$ .

**Theorem 4.3.1.** *Let  $l$  be as in Assumption 1.2.1. Choose  $w \in W$  and  $J \subseteq \Pi$  such that  $w(\Phi_0^+) = \Phi_J^+$ .*

- (a) *Suppose that  $l \nmid n+1$  when  $\Phi$  is of type  $A_n$  and  $l \neq 9$  when  $\Phi$  is of type  $E_6$ . Then as  $U_\zeta(\mathfrak{l}_J)$ -modules*

$$\text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^i(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})) = \begin{cases} \mathbb{C} & \text{if } i = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

- (b) *If  $\Phi$  is of type  $A_n$  with  $n+1 = l(m+1)$  and  $w$  is as defined in (4.8.1), then as  $U_\zeta(\mathfrak{l}_J)$ -modules*

$$\begin{aligned} &\text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^i(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})) \\ &= \begin{cases} \mathbb{C} & \text{if } i = \ell(w) \\ l\varpi_{t(m+1)} \oplus l\varpi_{(l-t)(m+1)} & \text{if } i = \ell(w) + (m+1)t(l-t) \\ & \text{for } 1 \leq t \leq (l-1)/2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c) If  $\Phi$  is of type  $E_6$  and  $l = 9$  (assuming that  $w$  and  $J$  are as in Appendix A.1), then as  $U_\zeta(\mathfrak{l}_J)$ -modules

$$\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^i(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})) = \begin{cases} l\varpi_1 \oplus l\varpi_6 & \text{if } i = 20 \\ \mathbb{C} & \text{if } i = \ell(w) = 8 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Since  $\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}))$  is a module for  $U_\zeta(\mathfrak{l}_J)$  on which  $u_\zeta(\mathfrak{l}_J)$  acts trivially, it is also a (finite dimensional—hence completely reducible) module for the universal enveloping algebra  $\mathbb{U}(\mathfrak{l}_J)$  (see Section 2.3). Thus, if  $\mathrm{Hom}_{u_\zeta(\mathfrak{l}_J)}((\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})) \neq 0$ , any  $U_\zeta(\mathfrak{l}_J)$ -composition factor will be of the form  $L_J(\nu)^{[1]}$  for a  $J$ -dominant weight  $\nu$ . In other words, there must be a nonzero  $U_\zeta(\mathfrak{l}_J)$ -homomorphism

$$(4.3.1) \quad (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* \otimes L_J(\nu)^{[1]} \rightarrow H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}).$$

Hence, the weight  $-w_{0,J}(w \cdot 0) + l\nu$  must appear in  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ , and so also in  $\Lambda_{\zeta,j}^\bullet$  by Proposition 2.9.1(b). The theorem now follows from Proposition 4.2.1 if each weight listed therein does indeed give rise to a non-trivial homomorphism as in (4.3.1).

By Proposition 2.9.1(a),

$$(4.3.2) \quad \sum_{n=0}^{\infty} (-1)^n \mathrm{ch} H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}) = \sum_{n=0}^{\dim(\mathfrak{u}_J)} (-1)^n \mathrm{ch} \Lambda_{\zeta,J}^n.$$

Since  $H^\bullet(\mathbb{U}(\mathfrak{u}_J), \mathbb{C})$  can be computed from  $\Lambda^\bullet(\mathfrak{u}_J^*)$  considered as a complex (with appropriately defined differential), we similarly have

$$(4.3.3) \quad \sum_{n=0}^{\dim(\mathfrak{u}_J)} (-1)^n \mathrm{ch} H^n(\mathbb{U}(\mathfrak{u}_J), \mathbb{C}) = \sum_{n=0}^{\dim(\mathfrak{u}_J)} (-1)^n \mathrm{ch} \Lambda^n(\mathfrak{u}_J^*).$$

Clearly,  $\mathrm{ch}_\zeta \Lambda_{\zeta,J}^n = \mathrm{ch} \Lambda^n(\mathfrak{u}_J^*)$ . Hence (4.3.2) and (4.3.3) give

$$(4.3.4) \quad \sum_{n=0}^{\infty} (-1)^n \mathrm{ch} H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}) = \sum_{n=0}^{\dim(\mathfrak{u}_J)} (-1)^n \mathrm{ch} H^n(\mathbb{U}(\mathfrak{u}_J), \mathbb{C}).$$

Let  ${}^J W = \{x \in W \mid x(\Phi^-) \cap \Phi^+ \subset \Phi^+ \setminus \Phi_J^+\}$  be the set of distinguished right  $W_J$ -coset representatives in  $W$ . By [W, Thm. 2.5.1.3],

$$(4.3.5) \quad \sum_{n=0}^{\dim(\mathfrak{u}_J)} (-1)^n \mathrm{ch} H^n(\mathbb{U}(\mathfrak{u}_J), \mathbb{C}) = \sum_{x \in {}^J W} (-1)^{\ell(x)} \mathrm{ch} L_J(-w_{0,J}(x \cdot 0)).$$

Also, (cf. [HK, Prop. 3.4.5])

$$\mathrm{ch}_\zeta \mathrm{ind}_{U_\zeta(\mathfrak{b}_{\mathfrak{l}_J})}^{U_\zeta(\mathfrak{l}_J)} (-w_{0,J}(x \cdot 0)) = \mathrm{ch} L_J(-w_{0,J}(x \cdot 0)).$$

Combining this with (4.3.4) and (4.3.5) gives

$$(4.3.6) \quad \sum_{n=0}^{\infty} (-1)^n \mathrm{ch} H^n(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}) = \sum_{x \in {}^J W} (-1)^{\ell(x)} \mathrm{ch} \mathrm{ind}_{U_\zeta(\mathfrak{b}_{\mathfrak{l}_J})}^{U_\zeta(\mathfrak{l}_J)} (-w_{0,J}(x \cdot 0)).$$

The weights  $\gamma$  in Proposition 4.2.1 are all “Steinberg weights” whose induced module is injective over  $U_\zeta(\mathfrak{l}_J)$ . Hence, they do not appear as a composition factor in any other induced module. For part (a), since only one such weight occurs, it gives rise to a composition factor on the right-hand side of (4.3.6) which cannot be canceled out by any other factor. Hence it appears as well on the left-hand

side which completes the proof. For parts (b) and (c), while multiple “Steinberg weights” appear, these weights are all distinct and hence give rise to distinct composition factors on the right-hand side of (4.3.6) which cannot cancel each other out. Hence, they all appear on the left-hand side as well.  $\square$

**Remark 4.3.2.** In those cases where the cohomology is two-dimensional, the differences of the weights are neither sums of positive roots nor sums of negative roots. Hence the isomorphisms in the theorem also hold as  $U_\zeta(\mathfrak{p}_J)$ -modules.

#### 4.4. Proof of Proposition 4.2.1

The remainder of Section 4 is devoted to proving Proposition 4.2.1. Note first of all that the weight  $-w_{0,J}(w \cdot 0)$  does appear in  $\Lambda_{\zeta,J}^\bullet$  in degree  $\ell(w)$  (cf. [GW, 7.3], [FP1, Prop. 2.2]). So the goal is to show that (in most cases) a weight  $\nu$  satisfying the hypothesis must in fact be zero. In Sections 4.5–4.8, the classical root systems will be considered. For these, we will mainly work with the  $\epsilon$ -basis that represents  $\Phi$  [Bo, p. 250] and  $\langle -, - \rangle$  will always denote the ordinary Euclidean inner product. In Section 4.5, we first show that  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in J$ . To do this, for each of the classical root systems  $X \in \{A_n, B_n, C_n, D_n\}$  we show the existence of an element  $\delta_X \in X$  with the following properties:

$$(4.4.1) \quad \max_\lambda \langle \lambda, \delta_X \rangle := \max\{\langle \lambda, \delta_X \rangle \mid \lambda \text{ a weight of } \Lambda_{\zeta,J}^\bullet\} = \langle -w_{0,J}(w \cdot 0), \delta_X \rangle \text{ and}$$

$$(4.4.2) \quad \langle \varpi_j, \delta_X \rangle > 0 \text{ for all fundamental weights } \varpi_j \text{ corresponding to } \alpha_j \in J.$$

When  $\Phi$  is of type  $A_n$  or  $C_n$ , one can choose  $\delta_X = \sum_{\alpha \in J} \alpha^\vee$  and the maximum value in (4.4.1) turns out to be simply  $(l-1)|J|$ .

Now assume that a  $\delta_X$  satisfying properties (4.4.1) and (4.4.2) exists and that  $-w_{0,J}(w \cdot 0) + l\nu$  is a  $J$ -dominant weight of  $\Lambda_{\zeta,J}^i$  for some  $J$ -dominant weight  $\nu$ . Then

$$\langle -w_{0,J}(w \cdot 0), \delta_X \rangle + l\langle \nu, \delta_X \rangle = \langle -w_{0,J}(w \cdot 0) + l\nu, \delta_X \rangle \leq \max_\lambda \langle \lambda, \delta_X \rangle = \langle -w_{0,J}(w \cdot 0), \delta_X \rangle.$$

This forces  $\nu$  to vanish on  $J$ . To show that  $\nu = 0$ , it remains to show that  $\langle \nu, \alpha^\vee \rangle = 0$  for  $\alpha \in \Pi \setminus J$ . In Section 4.6, this is shown for types  $B_n, C_n, D_n$ . Sections 4.7 and 4.8 are devoted to dealing with type  $A_n$  and showing how the “extra” weights arise in Proposition 4.2.1.

In Section 4.9, similar ideas along with direct computations will be used to deal with the exceptional root systems.

#### 4.5. The weight $\delta_X$

In this section, we construct a weight  $\delta_X$  which satisfies properties (4.4.1) and (4.4.2). We will first consider the case  $m \geq 2$  in Theorems 3.2.1 and 3.2.2.

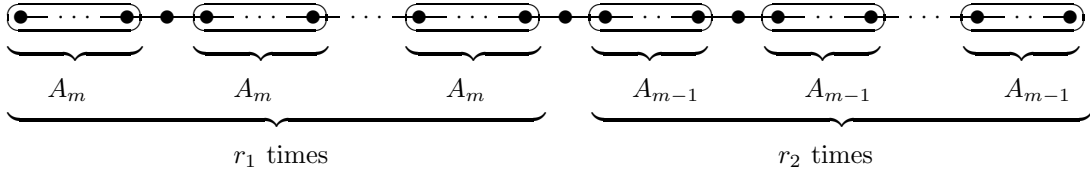
For  $\Phi$  of type  $A_n$  and

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{r_1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{r_2 \text{ times}};$$

we choose  $J$  in the “natural” way, namely such that

$$\Pi \setminus J = \{\alpha_{t(m+1)} \mid 1 \leq t \leq r_1\} \cup \{\alpha_{r_1(m+1)+sm} \mid 1 \leq s \leq r_2 - 1\}$$

and  $w$  such that  $w(\Phi_0^+) = \Phi_J^+$ . The Dynkin diagram below illustrates our choice for  $J$ .



Let

$$\delta_A = (\underbrace{1, 0, \dots, 0, -1, \dots, 1, 0, \dots, 0, -1}_{r_1 \text{ times}}, \underbrace{1, 0, \dots, 0, -1, \dots, 1, 0, \dots, 0, -1}_{r_2 \text{ times}})$$

in the orthonormal basis describing  $\Phi$  in  $n + 1$ -dimensional Euclidean space  $\mathbb{E}$  [Bo, p. 250]. Note that the first  $r_1$ -groupings of  $(1, 0, \dots, 0, -1)$  have  $m + 1$ -components, while the last  $r_2$ -groupings of  $(1, 0, \dots, 0, -1)$  have  $m$ -components. In this case,  $\delta_A = \sum_{\alpha \in J} \alpha^\vee$ , and evidently  $\langle \varpi_j, \delta_A \rangle > 0$  for all  $\varpi_j$  corresponding to simple roots in  $J$  (property (4.4.2)).

To show property (4.4.1), let  $\lambda$  be a weight of  $\Lambda_{\zeta, J}^i$ . Then  $\lambda$  is a sum of distinct positive roots not in  $\Phi_J^+$ . If  $\beta$  is a positive root then  $\langle \beta, \delta_A \rangle = 0, \pm 1, \pm 2$  (i. e.  $\beta = \epsilon_i - \epsilon_j$  with  $i < j$ ). Set

$$A[t] = \{\beta \in \Phi^+ \setminus \Phi_J^+ \mid \langle \beta, \delta_A \rangle = t\}.$$

A quick count shows that

$$\begin{aligned} |A[2]| &= \frac{(r_1 + r_2 - 1)(r_1 + r_2)}{2} \\ |A[1]| &= (m - 1)(r_1 + r_2 - 1)r_1 + (m - 2)(r_1 + r_2 - 1)r_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \max_\lambda \langle \lambda, \delta_A \rangle &= 2|A[2]| + |A[1]| \\ &= (r_1 + r_2 - 1)(mr_1 + (m - 1)r_2) \\ &= (r_1 + r_2 - 1)|J|. \end{aligned}$$

According to Theorem 3.2.1, we can set  $r_1 = s + 1$  and  $r_2 = l - s - 1$ . Consequently,

$$\max_\lambda \langle \lambda, \delta_A \rangle = (l - 1)|J| = \langle -w_{0, J}(w \cdot 0), \delta_A \rangle$$

because  $A_n$  is simply laced.

For the other classical Lie algebras, let  $\Phi$  be of type  $X_n$  where  $X = B, C, D$  with

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \dots \times A_m}_{r_1 \text{ times}} \times \underbrace{A_{m-1} \times \dots \times A_{m-1}}_{r_2 \text{ times}} \times X_q.$$

Again the “natural” choice for  $J$ , is such that

$$\Pi \setminus J = \{\alpha_{t(m+1)} \mid 1 \leq t \leq r_1\} \cup \{\alpha_{r_1(m+1)+sm} \mid 1 \leq s \leq r_2\}$$

with

$$\delta_X = (\underbrace{1, 0, \dots, 0, -1, \dots, 1, 0, \dots, 0, -1}_{r_1 \text{ times}}, \underbrace{1, 0, \dots, 0, -1, \dots, 1, 0, \dots, 0, -1}_{r_2 \text{ times}}, 1, 0, \dots, 0).$$

Notice that in all cases  $-w_{0, J}\delta_X = \delta_X$ . By using [GW, p. 102], one can verify that  $\langle \varpi_j, \delta_X \rangle > 0$  for all  $\varpi_j$  that correspond to simple roots in  $J$ . If  $\beta$  is a positive root in  $X_n$  then  $\langle \beta, \delta_X \rangle = 0, \pm 1, \pm 2$ , so

set  $X[t] = \{\beta \in \Phi^+ \setminus \Phi_J^+ \mid \langle \beta, \delta_X \rangle = t\}$ . By using our computations for type  $A_n$  with consideration of other positive roots in  $X_n$ , it follows that

$$|X[2]| = \begin{cases} (r_1 + r_2)^2 & X_n = B_n; \\ (r_1 + r_2)(r_1 + r_2 + 1) & X_n = C_n; \\ (r_1 + r_2)^2 & X_n = D_n \end{cases}$$

and

$$|X[1]| = \begin{cases} 2(r_1 + r_2)(r_1(m-1) + r_2(m-2)) + (2q-1)(r_1 + r_2) & X_n = B_n; \\ 2(r_1 + r_2)(r_1(m-1) + r_2(m-2)) + 2(q-1)(r_1 + r_2) & X_n = C_n; \\ 2(r_1 + r_2)(r_1(m-1) + r_2(m-2)) + 2(q-1)(r_1 + r_2) & X_n = D_n. \end{cases}$$

Hence,

$$\begin{aligned} \max_\lambda \langle \lambda, \delta_X \rangle &= 2|X[2]| + |X[1]| \\ &= \begin{cases} 2(r_1 + r_2)(r_1 m + r_2(m-1)) + (2q-1)(r_1 + r_2) & X_n = B_n; \\ 2(r_1 + r_2)(r_1 m + r_2(m-1)) + 2q(r_1 + r_2) & X_n = C_n; \\ 2(r_1 + r_2)(r_1 m + r_2(m-1)) + 2(q-1)(r_1 + r_2) & X_n = D_n. \end{cases} \end{aligned}$$

On the other hand, by using the expression of  $\rho$  for the classical Lie algebras in terms of the  $\epsilon$ -basis (see [GW, p. 107]), one sees that

$$\langle -w_{0,J}(w \cdot 0), \delta_X \rangle = \begin{cases} (l-1)(r_1 m + r_2(m-1)) + (l-1)(q - \frac{1}{2}) & X_n = B_n; \\ (l-1)(r_1 m + r_2(m-1)) + (l-1)(q) & X_n = C_n; \\ (l-1)(r_1 m + r_2(m-1)) + (l-1)(q-1) & X_n = D_n. \end{cases}$$

In Theorems 3.2.1 and 3.2.2,  $r_1 + r_2 = \frac{l-1}{2}$  for all root systems  $\Phi$  of types  $B_n$ ,  $C_n$ , or  $D_n$ , and so  $\max_\lambda \langle \lambda, \delta_X \rangle = \langle -w_{0,J}(w \cdot 0), \delta_X \rangle$  as desired.

Next we consider the case  $m = 1$ . Again let  $\Phi$  be of type  $X_n$  where  $X = A, B, C, D$  with

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_1 \times \cdots \times A_1}_{r \text{ times}} & \text{for } X_n = A_n, C_n, \text{ or } D_n. \\ \underbrace{A_1 \times \cdots \times A_1}_{r \text{ times}} \times A_1 & \text{for } X_n = B_n. \end{cases}$$

Here we choose  $J = \{\alpha_{2t-1} \mid 1 \leq t \leq r\}$  and  $J = \{\alpha_{2t-1} \mid 1 \leq t \leq r\} \cup \{\alpha_n\}$ , respectively. Let

$$\delta_X = \begin{cases} (\underbrace{(1, -1, \dots, 1, -1, 0, \dots, 0)}_{r \text{ times}}, \underbrace{0, \dots, 0}_{z \text{ times}}) & \text{for } X_n = A_n, C_n, \text{ or } D_n; \\ (\underbrace{(1, -1, \dots, 1, -1, 0, \dots, 0, 1)}_{r \text{ times}}, \underbrace{0, \dots, 0}_{z \text{ times}}) & \text{for } X_n = B_n. \end{cases}$$

As above, one can verify that  $\langle \varpi_j, \delta_X \rangle > 0$  for all  $\varpi_j$  that correspond to simple roots in  $J$ . Also, we conclude that

$$|X[2]| = \begin{cases} \frac{r(r-1)}{2} & X_n = A_n; \\ r^2 & X_n = B_n, C_n; \\ r(r-1) & X_n = D_n. \end{cases}$$

and

$$|X[1]| = \begin{cases} rz & X_n = A_n; \\ 2rz & X_n = C_n, D_n; \\ 2rz + r + z & X_n = B_n. \end{cases}$$

Hence,

$$\max_{\lambda} \langle \lambda, \delta_X \rangle = 2|X[2]| + |X[1]| = \begin{cases} r(r+z-1) & X_n = A_n; \\ (r+\frac{1}{2})(2r+2z) & X_n = B_n; \\ r(2r+2z) & X_n = C_n; \\ r(2(r+z-1)) & X_n = D_n. \end{cases}$$

On the other hand, by using the expression of  $\rho$  for the classical Lie algebras in terms of the  $\epsilon$ -basis (see [GW, p. 107]) one sees that

$$\langle -w_{0,J}(w \cdot 0), \delta_X \rangle = \begin{cases} r(l-1) & X_n = A_n, C_n, D_n; \\ (r+\frac{1}{2})(l-1) & X_n = B_n. \end{cases}$$

By Theorems 3.2.1 and 3.2.2,  $r$  is the number of copies of  $A_1$  and  $z = n+1-2r$  (respectively,  $n-2r-1, n-2r$ ) in type  $A_n$  (respectively,  $B_n, C_n$  or  $D_n$ ). One obtains that

$$r+z = \begin{cases} l & X_n = A_n; \\ \frac{l-1}{2} & X_n = B_n, C_n; \\ \frac{l+1}{2} & X_n = D_n. \end{cases}$$

Again,  $\max_{\lambda} \langle \lambda, \delta_X \rangle = \langle -w_{0,J}(w \cdot 0), \delta_X \rangle$ .

#### 4.6. Types $B_n, C_n, D_n$

In this section  $\Phi$  is always of type  $B_n, C_n$  or  $D_n$ . Under this assumption we show that the only weight  $\nu$  satisfying the hypothesis of Proposition 4.2.1 is the zero weight. Note that our restriction on the root systems and  $l$  being odd implies that  $\gcd(l, (X : Q)) = 1$ . For any such weight  $\nu$  we observe that  $l\nu \in Q$  because  $-w_{0,J}(w \cdot 0) + l\nu$  is a weight of  $\Lambda_{\zeta,J}^i$ . It follows that  $\nu \in Q$ .

Our results in Section 4.5 show that both  $-w_{0,J}(w \cdot 0)$  and  $-w_{0,J}(w \cdot 0) + l\nu$  consist of a sum of all the roots in  $X[1] \cup X[2]$  with some additional terms involving roots in  $X[0]$ . Therefore,  $l\nu$  is a sum of distinct roots in  $X[0] \cup -X[0]$ .

We express  $\delta_X$  and  $\nu$  in the  $\epsilon$ -basis as  $\delta_X = \sum_{i=1}^n \delta_{X,i} \epsilon_i$  and  $\nu = \sum_{i=1}^n \nu_i \epsilon_i$ , respectively. Notice that  $\nu \in Q$  implies that  $\nu \in \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$ . Define the following sets

$$(4.6.1) \quad S_{(a,b)} = \{(i,j) : \delta_{X,i} = a, \delta_{X,j} = b, \text{ and } i < j\} \text{ and } S_{(a)} = \{i : \delta_{X,i} = a\}.$$

The case when  $\Phi$  is of type  $B_n$  will be discussed in detail. The verification that  $\nu = 0$  for the cases when  $\Phi$  is of type  $C_n$  and  $D_n$  are left to the reader.

Any positive root of the form  $\epsilon_i - \epsilon_j$  in  $B[0]$  satisfies  $(i,j) \in S_{(0,0)} \cup S_{(1,1)} \cup S_{(-1,-1)}$ , any positive root of the form  $\epsilon_i + \epsilon_j$  in  $B[0]$  satisfies  $(i,j) \in S_{(0,0)} \cup S_{(1,-1)} \cup S_{(-1,1)}$  and any positive root of the form  $\epsilon_i$  in  $B[0]$  satisfies  $i \in S_{(0)}$ . Using (4.6.1), we can express

$$\begin{aligned} l\nu &= \sum_{(i,j) \in S_{(1,1)}} m_{i,j}(\epsilon_i - \epsilon_j) + \sum_{(i,j) \in S_{(-1,-1)}} \tilde{m}_{i,j}(\epsilon_i - \epsilon_j) \\ &+ \sum_{(i,j) \in S_{(1,-1)}} n_{i,j}(\epsilon_i + \epsilon_j) + \sum_{(i,j) \in S_{(-1,1)}} \tilde{n}_{i,j}(\epsilon_i + \epsilon_j) \\ &+ \sum_{(i,j) \in S_{(0,0)}} p_{i,j}(\epsilon_i - \epsilon_j) + \sum_{(i,j) \in S_{(0,0)}} \tilde{p}_{i,j}(\epsilon_i + \epsilon_j) + \sum_{i \in S_{(0)}} q_i \epsilon_i \end{aligned}$$

with  $m_{i,j}, \tilde{m}_{i,j}, n_{i,j}, \tilde{n}_{i,j}, q_i, p_{i,j}, \tilde{p}_{i,j} = 0, \pm 1$ .

The above expression shows that if  $\delta_{B,i} = 1$  then  $l\nu_i = \sum_j (m_{i,j} + n_{i,j} + m_{j,i} + \tilde{n}_{j,i})$ . For fixed  $i$ , it follows from Theorem 3.2.1 that the number of pairs of the form  $(i,j)$  together with those of form

$(j, i)$  in  $S_{(1,1)}$  are less than  $(l+1)/2$ . A similar counting argument shows that the number of pairs of the form  $(i, j)$  in  $S_{(1,-1)}$  together with the number of pairs of the form  $(j, i)$  in  $S_{(-1,1)}$  are less than  $(l-1)/2$ . Therefore,  $|\sum_j (m_{i,j} + n_{i,j} + m_{j,i} + \tilde{n}_{j,i})| \leq (l-1)$  and  $\delta_{B,i} = 1$  implies  $\nu_i = 0$ . Similarly, one can argue that  $\delta_{B,i} = -1$  implies  $\nu_i = 0$ .

Any simple root  $\epsilon_i - \epsilon_{i+1} \in \Pi \setminus J$  satisfies either  $\delta_{B,i} = -1$  and  $\delta_{B,i+1} = 1$  or  $\delta_{B,i} = 0$  and  $\delta_{B,i+1} = 0$ . It follows from above that, for any  $\alpha \in \Pi \setminus J$ ,

$$\langle l\nu, \alpha^\vee \rangle = \left\langle \sum_{i \in S_{(0)}} q_i \epsilon_i + \sum_{(i,j) \in S_{(0,0)}} p_{i,j} (\epsilon_i - \epsilon_j) + \sum_{(i,j) \in S_{(0,0)}} \tilde{p}_{i,j} (\epsilon_i + \epsilon_j), \alpha^\vee \right\rangle.$$

For  $m > 1$ , there are no roots in  $\Pi \setminus J$  with  $\delta_{B,i} = 0$  and  $\delta_{B,i+1} = 0$  and the inner product on the right-hand side vanishes. Since  $\nu$  vanishes on  $J$ , one concludes  $\nu = 0$ , as desired.

Assume  $m = 1$  and let  $i$  be such that  $\delta_{B,i} = 0$ , then  $l\nu_i = q_i + \sum_j (p_{i,j} + \tilde{p}_{i,j} + p_{j,i} + \tilde{p}_{j,i})$ . Theorem 3.2.1 implies that for fixed  $i$  there are less than  $(l-1)/2$  pairs of the form  $(i, j)$  or  $(j, i)$  in  $S_{(0,0)}$ . It follows that  $l|\nu_i| = |q_i + \sum_j (p_{i,j} + \tilde{p}_{i,j} + p_{j,i} + \tilde{p}_{j,i})| < l$ . This forces  $\nu = 0$ .

#### 4.7. Type $A_n$

In this (and the next) section  $\Phi$  is always of type  $A_n$  with simple roots  $\alpha_1, \dots, \alpha_n$ . We will show that a weight  $\nu$  satisfying the hypothesis of Proposition 4.2.1 equals the zero weight unless  $l$  divides  $n+1$ . For  $\mu \in Q$ ,  $\mu_i$  always denotes the coefficient of  $\mu$  in its expansion in terms of the  $\epsilon$ -basis (i. e.,  $\mu = \sum_{i=1}^{n+1} \mu_i \epsilon_i$ ).

First consider the case when  $m > 1$  in Theorem 3.2.1. Let

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \dots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \dots \times A_{m-1}}_{l-s-1 \text{ times}}$$

where  $n = lm + s$  and  $0 \leq s \leq l-1$ .

We choose  $J$  as in Section 4.5 and fix a particular  $w \in W$  with  $w(\Phi_0^+) = \Phi_J^+$  as follows. Partition the set  $\{\frac{n}{2}, \frac{n}{2} - 1, \dots, -\frac{n}{2} + 1, -\frac{n}{2}\}$ , i. e., the set of coordinates of  $\rho$  in the  $\epsilon$ -basis, into its congruence classes modulo  $l$  and order each congruence class in decreasing order. Then we order the congruence classes according to the size of their largest element from highest to lowest. The resulting array is the coordinate vector of a  $W$ -conjugate of  $\rho$ . We denote this conjugate by  $w\rho$  and  $w \in W$  denotes the unique permutation that sends  $\rho$  to  $w\rho$ . If we identify the Weyl group with the symmetric group in  $n+1$  letters, then  $w$  can be described as follows.

$$(4.7.1) \quad \text{For } 1 \leq i \leq n+1 \text{ define } s_i, t_i \text{ via } i-1 = s_i l + t_i \text{ with } 0 \leq t_i \leq l-1.$$

$$(4.7.2) \quad \text{Then } w(\epsilon_i) = \epsilon_{w(i)} \text{ where } w(i) = \begin{cases} t_i(m+1) + s_i + 1 & \text{if } 0 \leq t_i \leq s \\ t_i m + s_i + s + 2 & \text{if } s+1 \leq t_i \leq l-1. \end{cases}$$

$$(4.7.3) \quad \text{Moreover, } w_{0,J} w(i) = \begin{cases} (t_i+1)(m+1) - s_i & \text{if } 0 \leq t_i \leq s \\ (t_i+1)m - s_i + s + 1 & \text{if } s+1 \leq t_i \leq l-1. \end{cases}$$

For any  $u \in W$  define  $Q(u) := \{\alpha \in \Phi^+ \mid u\alpha \in \Phi^-\}$ . Using (4.7.1) and (4.7.2) we find that [GW, 7.3]

$$(4.7.4) \quad Q(w) = \{\epsilon_i - \epsilon_j \mid s_i < s_j, t_i > t_j\} \text{ and } w \cdot 0 = \sum_{\{\gamma \in Q(w)\}} -w(\gamma).$$

Define

$$f(i) = \begin{cases} w_{0,J} w(lm + i) = 1 + (i-1)(m+1) & \text{if } 1 \leq i \leq s+1 \\ w_{0,J} w(l(m-1) + i) = 1 + (i-1)m + (s+1) & \text{if } s+2 \leq i \leq l, \end{cases}$$

and

$$(4.7.5) \quad g(i) = \begin{cases} w_{0,J}w(i) = i(m+1) & \text{if } 1 \leq i \leq s+1 \\ w_{0,J}w(i) = im + (s+1) & \text{if } s+2 \leq i \leq l. \end{cases}$$

Next set  $\delta^i = \epsilon_{f(i)} - \epsilon_{g(i)}$  for  $1 \leq i \leq l$ . Then  $\delta_A = \sum_{i=1}^l \delta^i = \sum_{i=1}^l \epsilon_{f(i)} - \epsilon_{g(i)}$ .

The  $\alpha_{g(i)} = \epsilon_{g(i)} - \epsilon_{f(i+1)}$  are precisely the simple roots contained in  $\Pi$  but not in  $J$ .

Using the notation introduced in (4.6.1), we partition  $A[0]$  into the following subsets:

$$\begin{aligned} R_{(0,0)}^+ &= \{\epsilon_i - \epsilon_j \in A[0] \mid (i, j) \in S_{(0,0)}\}, \\ R_{(1,1)}^+ &= \{\epsilon_i - \epsilon_j \in A[0] \mid (i, j) \in S_{(1,1)}\} = \{\epsilon_{f(i)} - \epsilon_{f(j)} \mid 1 \leq i < j \leq l\}, \\ R_{(-1,-1)}^+ &= \{\epsilon_i - \epsilon_j \in A[0] \mid (i, j) \in S_{(-1,-1)}\} = \{\epsilon_{g(i)} - \epsilon_{g(j)} \mid 1 \leq i < j \leq l\}. \end{aligned}$$

In addition we set  $R_{(a,a)} = R_{(a,a)}^+ \cup -R_{(a,a)}^+, a \in \{-1, 0, 1\}$ . Notice that both sets  $R_{(1,1)}$  and  $R_{(-1,-1)}$  form root systems of type  $A_{l-1}$  with simple roots

$$(4.7.6) \quad \beta_i := \epsilon_{f(i)} - \epsilon_{f(i+1)} \text{ and } \tau_i := \epsilon_{g(i)} - \epsilon_{g(i+1)},$$

respectively.

Next define

$$S^+ := \{\epsilon_{f(i)} - \epsilon_{f(j)} \mid 1 \leq i \leq s+1, s+2 \leq j \leq l\} \text{ and } S := S^+ \cup -S^+.$$

Then it follows from (4.7.3) through (4.7.5) that

$$(4.7.7) \quad R_{(-1,-1)}^+ \cap w_{0,J}w(Q(w)) = \emptyset \text{ and } R_{(1,1)}^+ \cap w_{0,J}w(Q(w)) = S^+.$$

One concludes that the weight  $-w_{0,J}(w \cdot 0)$  is the sum of all roots in  $A[1] \cup A[2]$  together with certain roots in  $R_{(0,0)}^+$  and the roots in  $S^+$ . The elements of  $S^+$  can also be characterized as those roots in  $R_{(1,1)}^+$  that contain  $\beta_{s+1}$ . It is important to note that no roots of  $R_{(-1,-1)}^+$  contribute to  $-w_{0,J}(w \cdot 0)$ .

Next assume that  $\lambda$  is a weight of  $\Lambda_{\zeta,J}^\bullet$  such that  $\langle \lambda + w_{0,J}(w \cdot 0), \alpha \rangle = 0$  for all  $\alpha \in J$ . Set  $\nu = \lambda + w_{0,J}(w \cdot 0)$ . Using the  $\beta$ -basis of  $R_{(1,1)}$  and the  $\gamma$ -basis of  $R_{(-1,-1)}$ , we express  $\nu$  in the form

$$(4.7.8) \quad \nu = \sum_{i=1}^{l-1} k_i \beta_i + \sum_{i=1}^{l-1} l_i \tau_i + \sum_{\eta \in R_{(0,0)}^+} m_\eta \eta.$$

Since  $\nu$  is the zero weight when restricted to  $J$ , one observes that  $\langle \nu, \delta^i \rangle = 0$  for  $1 \leq i \leq l$ . Since  $\langle \nu, \delta^1 \rangle = 0$ ,  $k_1 - l_1 = 0$  and, inductively, it follows from  $\langle \nu, \delta^i \rangle = 0$  that  $k_i = l_i$  for  $1 \leq i \leq l-1$ . Moreover, it follows from (4.7.7) that all  $k_i$  and  $l_i$  are nonnegative. One concludes that  $\nu$  is a sum of distinct roots in  $R_{(0,0)}$  together with distinct roots in  $R_{(-1,-1)}^+$  and in  $R_{(1,1)}^+ \setminus S^+$ .

Finally, assume that  $\lambda_1$  and  $\lambda_2$  are two weights of  $\Lambda_{\zeta,J}^\bullet$  such that  $\langle \lambda_i + w_{0,J}(w \cdot 0), \alpha \rangle = 0$  for all  $\alpha \in J$ . For example  $\lambda_1$  could be of the form  $-w_{0,J}(u \cdot 0)$  for some  $u \neq w$  with  $u(\Phi_0^+) = \Phi_J^+$  and  $\lambda_2$  could be equal to  $-w_{0,J}(u \cdot 0) + l\nu$ , where  $\nu$  is the zero weight when restricted to  $J$ . It follows from our above arguments that  $\lambda_2 - \lambda_1$  is a sum of distinct roots in  $R_{(-1,-1)} \cup R_{(1,1)} \setminus S \cup R_{(0,0)}$ . The elements in  $R_{(1,1)} \setminus S$  form a root system of type  $A_s \times A_{l-s-2}$ , spanned by the roots  $\{\beta_1, \dots, \beta_s\} \cup \{\beta_{s+2}, \dots, \beta_{l-1}\}$ , as defined in (4.7.6). We can decompose  $\lambda_2 - \lambda_1 = \gamma_1 + \gamma_2 + \gamma_3$  where the support of  $\gamma_1$  lies entirely in the type  $A_s$  component of  $R_{(1,1)} \setminus S$ , the support of  $\gamma_2$  lies entirely in the type  $A_{l-s-2}$  component of  $R_{(1,1)} \setminus S$ , and the support of  $\gamma_3$  lies entirely in  $R_{(-1,-1)} \cup R_{(0,0)}$ .



Next observe that  $\alpha_{g(i)} = \beta_i - \delta^i$ . It follows that  $\langle \lambda_2 - \lambda_1, \alpha_{g(i)} \rangle = \langle \lambda_2 - \lambda_1, \beta_i \rangle = \langle \gamma_1 + \gamma_2, \beta_i \rangle$ . The inner product of  $\lambda_2 - \lambda_1$  with the roots in  $\Pi \setminus J$  is then given by the following:

$$(4.7.9) \quad \langle \lambda_2 - \lambda_1, \alpha_{g(i)} \rangle = \begin{cases} \langle \gamma_1, \beta_i \rangle & \text{if } 1 \leq i \leq s, \\ \langle \gamma_1 + \gamma_2, \beta_i \rangle & \text{if } i = s+1, \\ \langle \gamma_2, \beta_i \rangle & \text{if } s+2 \leq i \leq l-1. \end{cases}$$

Since  $\gamma_1$  is a sum of distinct roots of a root system of type  $A_s$  a direct computation shows that  $|\langle \gamma_1, \beta_i \rangle| \leq s+1$ . Similarly,  $|\langle \gamma_2, \beta_i \rangle| \leq l-s-1$ . Now  $\lambda_2 - \lambda_1 \in lX$  implies that either  $\lambda_1 = \lambda_2$  or  $l = s+1$ . Hence, Proposition 4.2.1 holds for type  $A_n$  as long as  $l$  does not divide  $n+1$ .

Suppose now that  $m = 1$ . Then

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_1 \times \cdots \times A_1}_{s+1 \text{ times}}.$$

Essentially the same argument as above works here as well. We highlight some of the differences and leave the details to the interested reader. The definition of  $w$  holds as above with  $m = 1$ . Define  $f(i)$ ,  $g(i)$ ,  $\delta^i$ ,  $\delta_A$ , and  $\alpha_{g(i)}$  just as above with  $m = 1$ . Note that for  $i \geq s+2$ ,  $f(i) = g(i)$  and  $\delta^i = 0$ . In the definitions of  $R_{(1,1)}^+$  and  $R_{(-1,-1)}^+$ , we have  $1 \leq i < j \leq s+1$ , and the root systems  $R_{(1,1)}$  and  $R_{(-1,-1)}$  are of type  $A_s$ , while  $S^+ = \emptyset$ . For  $s+2 \leq i \leq l-1$ , the  $\beta_i = \tau_i$  form a basis for the root system  $R_{(0,0)}$  of type  $A_{l-s-2}$ . The equivalent of (4.7.7) is now

$$R_{(a,a)}^+ \cap w_{0,J}w(Q(w)) = \emptyset \text{ where } a \in \{1, 0, -1\}.$$

In (4.7.8), the index  $i$  should run from  $i = 1$  to  $i = s$ . Next we consider  $\lambda_2 - \lambda_1$ , as defined above. We decompose  $\lambda_2 - \lambda_1 = \gamma_1 + \gamma_2 + \gamma_3$  with the support of  $\gamma_1$  in  $R_{(1,1)}$ , the support of  $\gamma_2$  in  $R_{(0,0)}$  and the support of  $\gamma_3$  in  $R_{(-1,-1)}$ . Then equation (4.7.9) remains valid. Hence one obtains the same conclusion.

#### 4.8. Type $A_n$ with $l$ dividing $n+1$

Let  $\Phi$  be a root system of type  $A_n$ . We begin with the special case  $l = n+1 = h$ . Here  $J = \emptyset$ . We will make use of the following Lemma.

**Lemma 4.8.1.** *Let  $\Phi$  be of type  $A_n$ ,  $l = n+1$  and  $\nu \in X$ . The weight  $l\nu$  appears in  $\Lambda_{\zeta, \emptyset}^\bullet$  if and only if  $\nu = \varpi_i$  for some  $0 \leq i \leq n$ , where  $\varpi_0 = 0$ .*

PROOF. Assume that  $l\nu = (n+1)\nu$  is a weight of  $\Lambda_{\zeta, \emptyset}^\bullet$ . It follows from the argument in [AJ, 2.2, 6.1] that  $\nu = u\varpi_i$ , for some  $u \in W$  and  $0 \leq i \leq n$ . Next assume that  $\nu = u\varpi_i \neq \varpi_i$ . Note that  $(n+1)\varpi_i = \sum_{j=1}^i j(n+1-i)\alpha_j + \sum_{j=i+1}^n i(n+1-j)\alpha_j$  and that  $(n+1)u\varpi_i = \sum_{j=1}^i [j(n+1-i) - q_j(n+1)]\alpha_j + \sum_{j=i+1}^n [i(n+1-j) - p_j(n+1)]\alpha_j$  for some  $q_j, p_j \geq 0$ . Since  $(n+1)u\varpi_i$  is a sum of positive roots it follows that  $q_1 = 0$ , while  $u\varpi_i \neq \varpi_i$  implies  $q_i \geq 1$ . Therefore, there exists a  $j$  with  $1 \leq j < i$  such that  $q_j = 0$  and  $q_{j+1} \geq 1$ . Since  $(n+1)u\varpi_i \in \Lambda_{\zeta, \emptyset}^\bullet$ , it is a sum of *distinct* positive roots. From the preceding decomposition into simple roots and the assumption on  $j$ , this sum includes precisely  $j(n+1-i)$  distinct roots that contain the simple root  $\alpha_j$ . However, this sum contains at most  $(j+1)(n+1-i) - (n+1)$  distinct roots that contain  $\alpha_{j+1}$  and hence at most  $(j+1)(n+1-i) - (n+1)$  distinct roots that contain  $\alpha_j + \alpha_{j+1}$ . On the other hand, there are only  $j$  distinct roots that contain  $\alpha_j$  but not  $\alpha_{j+1}$ . But  $(j+1)(n+1-i) - (n+1) + j = j(n+1-i) - i + j < j(n+1-i)$ , which is a contradiction.

The weight  $l\varpi_i$  is precisely the sum of all positive roots containing  $\alpha_i$ . Hence, it is a weight of  $\Lambda_{\zeta, \emptyset}^\bullet$ .  $\square$

Assume throughout the remainder of this section that  $l$  divides  $n + 1$ . We continue to identify the Weyl group with the symmetric group in  $n + 1$  letters. Then  $l = s + 1$  and the definition of  $w$  given in (4.7.1) and (4.7.2) can be simplified to

$$(4.8.1) \quad w(\epsilon_i) = \epsilon_{w(i)} \text{ where } w(i) = t_i(m + 1) + s_i + 1.$$

We now follow the arguments used in [AJ, 6.2]. Recall that  $n + 1 = (m + 1)l$ . We define the element  $\sigma \in W$  as follows:

$$\sigma = (1, 2, \dots, l)(l + 1, l + 2, \dots, 2l) \cdots (ml + 1, ml + 2, \dots, (m + 1)l).$$

Direct computation shows that  $w(\sigma^t \cdot 0) = -l\varpi_{g(t)}$  for  $1 \leq t \leq l - 1$ . Setting  $\varpi_{g(0)} = 0$  yields

$$(4.8.2) \quad w \cdot 0 = w\sigma^t \cdot 0 + l\varpi_{g(t)} \text{ for } 0 \leq t \leq l - 1.$$

We find that

$$\begin{aligned} Q(w\sigma^t) &= \{\epsilon_i - \epsilon_j \mid s_i < s_j, \sigma^t(t_i + 1) > \sigma^t(t_j + 1)\} \\ &\cup \{\epsilon_i - \epsilon_j \mid s_i = s_j, t_i < t_j \text{ and } \sigma^t(t_i + 1) > \sigma^t(t_j + 1)\}. \end{aligned}$$

The cardinality of the first set in the above union is equal to the cardinality of  $Q(w)$  (see (4.7.4)) while the second set can be identified with  $Q(\sigma^t)$ . Using this decomposition of  $Q(w\sigma^t)$ , [GW, 7.3], and [AJ, 6.2(3)], we conclude that

$$(4.8.3) \quad \ell(w\sigma^t) = \ell(w) + \ell(\sigma^t) = \ell(w) + (m + 1)t(l - t).$$

Next, assume that  $l\nu - w_{0,J}(w \cdot 0)$  is a weight of  $\Lambda_{\zeta,J}^\bullet$  such that  $\langle l\nu, \alpha \rangle = 0$  for all  $\alpha \in J$ . The discussion in Section 4.7 shows that  $l\nu$  is a sum of distinct roots in  $R_{(-1,-1)}^+ \cup R_{(0,0)} \cup R_{(1,1)}^+$ . We can decompose  $\nu = \gamma_1 + \gamma_2$  where the support of  $\gamma_1$  lies entirely in  $R_{(1,1)}^+$  and the support of  $\gamma_2$  lies entirely in  $R_{(-1,-1)}^+ \cup R_{(0,0)}$ . As before, the inner product  $\langle l\nu, \alpha_{g(i)} \rangle = \langle l\nu, \beta_i \rangle = \langle l\gamma_1, \beta_i \rangle$  is completely determined by the contribution coming from  $R_{(1,1)}^+$ , the positive roots of a type  $A_{l-1}$  root system. Let  $\kappa_i$  denote the fundamental weight corresponding to the simple root  $\beta_i$  of the root system of type  $A_{l-1}$ . It follows from the above lemma that  $l\nu = l\kappa_i$ . Moreover, it follows from the construction that  $\kappa_i = \varpi_{g(i)}$ . Finally, by (4.8.2), for each  $1 \leq i \leq l - 1$ , we have

$$-w_{0,J}(w \cdot 0) + l\varpi_{g(i)} = -w_{0,J}(w\sigma^i \cdot 0),$$

and the latter weight is a weight of  $\Lambda_{\zeta,J}^\bullet$  (cf. [GW, 7.3], [FP1, Prop. 2.2]). Further, by (4.8.3), this lies in degree  $\ell(w\sigma^i) = \ell(w) + (m + 1)t(l - i)$ . Since  $g(i) = i(m + 1)$ , the result follows.

#### 4.9. Exceptional Lie algebras

In this section, we assume that the root system  $\Phi$  is of exceptional type. We show that if  $\nu$  satisfies the hypothesis of Proposition 4.2.1, then  $\nu = 0$  except in the case when  $\Phi$  is of type  $E_6$  and  $l = 9$ . Note that in the excluded case  $l$  is divisible by  $(X : \mathbb{Z}\Phi) = 3$ . Our goal is to show that if  $-w_{0,J}(w \cdot 0) + l\nu$  with  $\nu$  being  $J$ -dominant is a weight of  $\Lambda_{\zeta,J}^\bullet$ , then  $\nu = 0$ . For the exceptional Lie algebras, an explicit choice of  $w$  and  $J$  is listed in Appendix A.1. One can then explicitly compute the value of  $-w_{0,J}(w \cdot 0)$ . This was again done with the aid of MAGMA [BC, BCP] and the results are given in tables in Appendix A.2. The weights in the tables are listed with respect to the basis  $\{\varpi_1, \dots, \varpi_n\}$  of fundamental dominant weights.

Since the dimension of  $\Lambda_{\zeta,J}^\bullet$  is finite, with the aid of a computer, one could in principle compute all possible weights of  $\Lambda_{\zeta,J}^\bullet$  and compare them to  $-w_{0,J}(w \cdot 0)$  modulo  $l$ . For types  $F_4$  and  $G_2$ , this can readily be done and one finds that  $\nu = 0$ . For type  $E_n$ , the size of  $\Lambda_{\zeta,J}^\bullet$  is sufficiently large as to make the computations somewhat impractical on a typical desktop computer. As such, we present

an alternative approach which makes use of some of the ideas from the preceding sections on classical root systems to show directly that  $\nu = 0$  or reduce the computations to a more manageable number.

In what follows, let  $\delta^\vee = \sum_{\alpha \in J} \alpha^\vee$ . This is a slight abuse of notation since  $\delta^\vee$  may not equal  $(\sum_{\alpha \in J} \alpha)^\vee$  but should not lead to any confusion. Recall that  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1)|J|$ . As in Section 4.4, set  $\max_\lambda \langle \lambda, \delta^\vee \rangle := \max\{\langle \lambda, \delta^\vee \rangle \mid \lambda \text{ a weight of } \Lambda_{\zeta,J}^i\}$ . As with the classical cases, the key to showing that  $\nu = 0$  is that  $\max_\lambda \langle \lambda, \delta^\vee \rangle$  is in general “close” to  $(l-1)|J|$ .

To make this more precise, set

$$E[t] := \{\beta \in \Phi^+ \setminus \Phi_J^+ \mid \langle \beta, \delta^\vee \rangle = t\}.$$

Then we can decompose  $\Phi^+ \setminus \Phi_J^+$  as a disjoint union:

$$\Phi^+ \setminus \Phi_J^+ = E[< 0] \cup E[0] \cup E[> 0]$$

where

$$E[> 0] = \cup_{t>0} E[t] \text{ and } E[< 0] = \cup_{t<0} E[t].$$

That is, we separate the positive roots into those which give a positive, zero, or negative inner product with  $\delta^\vee$ . Since weights in  $\Lambda_{\zeta,J}^\bullet$  are composed of sums of distinct positive roots from  $\Phi^+ \setminus \Phi_J^+$ , clearly,

$$\max_\lambda \langle \lambda, \delta^\vee \rangle = \langle \sum_{\beta \in E[>0]} \beta, \delta^\vee \rangle.$$

For convenience, for the remainder of this section, set  $\lambda := \sum_{\beta \in E[>0]} \beta$ . One might think of  $\lambda$  as a conical representative of those weights having maximum inner product with  $\delta^\vee$ . If  $\sigma$  is a weight of  $\Lambda_{\zeta,J}^i$  with  $\langle \sigma, \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$ , then we would have  $\sigma = \lambda + z$  where  $z$  is a sum of distinct roots which lie in  $E[0]$ . With the aid of MAGMA the weight  $\lambda$  can be readily computed for a given  $l$ ,  $w$ , and  $J$ . For each relevant case, the weight  $\lambda$  is given in the tables in Appendix A.2.

Let  $x = -w_{0,J}(w \cdot 0) + l\nu$  be a  $J$ -dominant weight of  $\Lambda_{\zeta,J}^\bullet$ . Our goal is to show that (in all but one case) the only such weight that occurs is when  $\nu = 0$ . Recall that, as mentioned in Section 4.4, the weight  $-w_{0,J}(w \cdot 0)$  appears in  $\Lambda_{\zeta,J}^{\ell(w)}$ . To show that  $\nu = 0$ , we show that  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in \Pi$ . We separate this into two cases:  $\alpha \in J$  and  $\alpha \in \Pi \setminus J$ . The first case follows if it can be shown that

$$|\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle - \langle \lambda, \delta^\vee \rangle| < l.$$

The numbers  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  can be found by direct calculation. These values are given in the tables in Appendix A.2. We find that the desired inequality holds in all but one case (type  $E_8$  when  $l = 7$ ). In that one remaining case, a different argument will be used to show that  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in J$ .

We now outline the basic process for showing that  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in \Pi \setminus J$ . Note that there will be precisely one case (type  $E_6$  when  $l = 9$ ) when this fails to hold. Once it is known that  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in J$ , it follows that  $\langle \nu, \delta^\vee \rangle = 0$ . Hence  $\langle x, \delta^\vee \rangle = \langle -w_{0,J}(w \cdot 0) + l\nu, \delta^\vee \rangle = \langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1)|J|$ . By the definition of  $\lambda$ , we have  $\langle \lambda, \delta^\vee \rangle \geq \langle x, \delta^\vee \rangle = (l-1)|J|$ . The underlying idea of the proof is that  $\langle \lambda, \delta^\vee \rangle$  and  $\langle x, \delta^\vee \rangle$  are in many cases equal or in general differ by only a small amount. We express  $x$  in the form  $x = \lambda - a + b + z$  where  $a$ ,  $b$ , and  $z$  consist of (possibly empty) sums of distinct roots lying in  $E[> 0]$ ,  $E[< 0]$ , and  $E[0]$ , respectively. While  $z$  can consist of arbitrarily many elements from  $E[0]$ ,  $a$  and  $b$  are constrained to consist of a small number of elements from  $E[> 0]$  or  $E[< 0]$  depending upon how close  $\langle \lambda, \delta^\vee \rangle$  is to  $(l-1)|J|$ . In each case, this can be explicitly described.

Given  $\alpha \in \Pi \setminus J$ , by direct computation, one can find bounds  $A$  and  $B$  (integers) such that for an arbitrary linear combination  $z$  as above, one has  $A \leq \langle z, \alpha^\vee \rangle \leq B$ . In some cases, these bounds will be sufficient to conclude that  $\nu = 0$ . For such cases, the bounds will be given below. When that is not sufficient, using the expression of  $x$  as  $\lambda + a - b + z$ , and considering the possibilities for  $a$  and  $b$ ,

one can then obtain bounds  $A' \leq \langle x, \alpha^\vee \rangle \leq B'$  for each  $\alpha \in \Pi \setminus J$ . In those cases needed, the bounds are listed in Appendix A.2. On the other hand,

$$\langle x, \alpha^\vee \rangle = \langle -w_{0,J}(w \cdot 0) + l\nu, \alpha^\vee \rangle = \langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle + l\langle \nu, \alpha^\vee \rangle.$$

The value of  $\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$  can also be computed, and (as needed) is listed in Appendix A.2. Comparing this to the bounds on  $\langle x, \alpha^\vee \rangle$ , one often finds that  $\langle \nu, \alpha^\vee \rangle$  must be zero. When the bounds allow for a nonzero  $\nu$ , MAGMA is used to verify that no solutions exist (except in type  $E_6$  when  $l = 9$ ) by checking all possibilities for  $a$ ,  $b$ , and  $z$ . When needed for efficiency, the known bounds can be used to limit the possible choices for  $z$ . The basic details for each case are given below.

Type  $E_6$ :

$l = 11$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . For  $x = \lambda - a + b + z$  as above, we have

$$\begin{aligned} \langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle &= \langle x, \delta^\vee \rangle \\ &= \langle \lambda, \delta^\vee \rangle - \langle a, \delta^\vee \rangle + \langle b, \delta^\vee \rangle + \langle z, \delta^\vee \rangle \\ &= \langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle - \langle a, \delta^\vee \rangle + \langle b, \delta^\vee \rangle. \end{aligned}$$

By definition  $\langle a, \delta^\vee \rangle \geq 0$  and  $\langle b, \delta^\vee \rangle \leq 0$ , and so  $a$  and  $b$  must be empty. Hence  $x = \lambda + z$ . Since we are also assuming that  $x = -w_{0,J}(w \cdot 0) + l\nu$ , it follows that  $z$  would need to equal  $l\nu$ . However, for  $\alpha \in \Pi \setminus J$ , one finds that  $-4 \leq \langle z, \alpha^\vee \rangle \leq 6$ . Hence  $z$  cannot equal  $11\nu$  unless  $\nu = 0$ .

$l = 9$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by two. Also  $E[> 0] = E[1]$  and  $E[< 0] = E[-1]$ . So there are three ways we can express  $x$  in the form  $\lambda - a + b + z$ :

- (1)  $x = \lambda - x_1 - x_2 + z$  where  $x_1, x_2 \in E[1]$  ( $x_1 \neq x_2$ ),
- (2)  $x = \lambda - x_1 + y_1 + z$  where  $x_1 \in E[1]$  and  $y_1 \in E[-1]$ ,
- (3)  $x = \lambda + y_1 + y_2 + z$  where  $y_1, y_2 \in E[-1]$  ( $y_1 \neq y_2$ ).

Using MAGMA, we compute all such weights  $\lambda - a + b + z$  and check whether or not they can be equal to  $-w_{0,J}(w \cdot 0) + l\nu$  for a  $J$ -dominant weight  $\nu$ . In case (1), we find precisely one pair of elements in  $E[1]$  that works with  $z$  being an empty sum and  $\nu$  being zero. That is,  $-w_{0,J}(w \cdot 0)$  is a sum of eight distinct roots in  $E[1]$ . In case (2), no sums over  $E[0]$  work. In case (3), we find however two cases where  $\lambda$  plus two elements of  $E[-1]$  and eight elements of  $E[0]$  equals  $-w_{0,J}(w \cdot 0) + 9\nu$  for a  $J$ -dominant weight  $\nu$ . In one case  $\nu = \varpi_1$  and in the other  $\nu = \varpi_6$ . The reader should be aware that these weights give rise to the exceptions stated in Theorem 1.2.3(b)(iii).

$l = 7$ : Here  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$ . Arguing as in the  $l = 11$  case, it follows that  $x$  must be of the form  $x = \lambda + z$ . For  $\alpha \in \Pi \setminus J$ , one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. For each  $\alpha$ , in order to have  $x = -w_{0,J}(w \cdot 0) + 7\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 5$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by one. So there are two ways we can express  $x$  in the form  $\lambda - a + b + z$ :

- (1)  $x = \lambda - x_1 + z$  where  $x_1 \in E[1]$ ,
- (2)  $x = \lambda + y_1 + z$  where  $y_1 \in E[-1]$ .

Using MAGMA, we compute all possibilities and find that only one such expression gives  $x = -w_{0,J}(w \cdot 0) + l\nu$ . Specifically, this occurs in case (1) with  $z$  being the sum of a pair of elements from  $E[0]$ . Again,  $x = -w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

Type  $E_7$ :

$l = 17$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence,  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . However, for  $\alpha \in \Pi \setminus J$ , one finds that  $-9 \leq \langle z, \alpha^\vee \rangle \leq 10$ . Hence,  $z$  cannot equal  $17\nu$  unless  $\nu = 0$ .

$l = 15$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by two. Also  $E[> 0] = E[1]$  and  $E[< 0] = E[-1]$ . As in the type  $E_6$ ,  $l = 9$  case, there are three ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$ , one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. For each  $\alpha$ , in order to have  $x = -w_{0,J}(w \cdot 0) + 15\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 13$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . However, for  $\alpha \in \Pi \setminus J$ , one finds that  $-6 \leq \langle z, \alpha^\vee \rangle \leq 8$ . Hence  $z$  cannot equal  $13\nu$  unless  $\nu = 0$ .

$l = 11$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . However, for  $\alpha \in \Pi \setminus J$ , one finds that  $-4 \leq \langle z, \alpha^\vee \rangle \leq 6$ . Hence  $z$  cannot equal  $11\nu$  unless  $\nu = 0$ .

$l = 9$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by two. Also  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . If we express  $x$  in the form  $\lambda - a + b + z$ , note that  $a$  cannot involve any terms from  $E[3]$  and  $b$  cannot involve any terms from  $E[-3]$ . So there are five ways we can express  $x$  in the form  $\lambda - a + b + z$ :

- (1)  $x = \lambda - x_1 + z$  where  $x_1 \in E[2]$ ,
- (2)  $x = \lambda - x_1 - x_2 + z$  where  $x_1, x_2 \in E[1]$  ( $x_1 \neq x_2$ ),
- (3)  $x = \lambda - x_1 + y_1 + z$  where  $x_1 \in E[1]$  and  $y_1 \in E[-1]$ ,
- (4)  $x = \lambda + y_1 + y_2 + z$  where  $y_1, y_2 \in E[-1]$  ( $y_1 \neq y_2$ ),
- (5)  $x = \lambda + y_1 + z$  where  $y_2 \in E[-2]$ .

For  $\alpha \in \Pi \setminus J$ , one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. For each  $\alpha$ , in order to have  $x = -w_{0,J}(w \cdot 0) + 9\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 7$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by three. Also  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . So there are ten ways we can express  $x$  in the form  $\lambda - a + b + z$ . We leave the details to the interested reader. In this case, for  $\alpha \in \Pi \setminus J$ , the bounds on  $\langle x, \alpha^\vee \rangle$  allow for the possibility that  $\langle \nu, \alpha^\vee \rangle \neq 0$ . Using MAGMA, we compute all possibilities and find that only one such expression gives  $x = -w_{0,J}(w \cdot 0) + l\nu$ . Specifically, this occurs for an  $x$  of the form  $x = \lambda - a$  where  $a$  is a sum of three distinct roots in  $E[1]$  and  $b$  and  $z$  are empty. Again,  $x = -w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

$l = 5$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by three. Also  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . As in the  $l = 7$  case, there are ten ways we can express  $x$  in the form  $\lambda - a + b + z$ . Here  $\Pi \setminus J = \{\alpha_4\}$  and one finds that  $-10 \leq \langle x, \alpha_4^\vee \rangle \leq -6$ . Since  $\langle -w_{0,J}(w \cdot 0), \alpha_4^\vee \rangle = -9$ , in order to have  $x = -w_{0,J}(w \cdot 0) + 5\nu$ , we must have  $\langle \nu, \alpha_4^\vee \rangle = 0$ .

Type  $E_8$ :

$l = 29$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . However, for  $\alpha \in \Pi \setminus J$ , one finds that  $-16 \leq \langle z, \alpha^\vee \rangle \leq 18$ . Hence  $z$  cannot equal  $29\nu$  unless  $\nu = 0$ .

$l = 27$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by two. Also  $E[> 0] = E[1]$  and  $E[< 0] = E[-1]$ . As in the type  $E_6$ ,  $l = 9$  case, there are three ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 27\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 25$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by four. Also  $E[> 0] = E[1]$  and  $E[< 0] = E[-1]$ . So there are five ways we can express  $x$  in the form  $\lambda - a + b + z$ :

- (1)  $x = \lambda - x_1 - x_2 - x_3 - x_4 + z$  where  $x_i \in E[1]$  (distinct),
- (2)  $x = \lambda - x_1 - x_2 - x_3 + y_1 + z$  where  $x_i \in E(1)$  (distinct) and  $y_1 \in E[-1]$ ,
- (3)  $x = \lambda - x_1 - x_2 + y_1 + y_2 + z$  where  $x_i \in E[1]$  (distinct) and  $y_i \in E[-1]$  (distinct),

- (4)  $x = \lambda - x_1 + y_1 + y_2 + y_3 + z$  where  $x_1 \in E[1]$  and  $y_i \in E[-1]$  (distinct),
- (5)  $x = \lambda + y_1 + y_2 + y_3 + y_4 + z$  where  $y_i \in E[-1]$  (distinct).

For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 25\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 23$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$ . Hence  $x = \lambda + z$ . For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 23\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 21$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by four. Also  $E[> 0] = E[1] \cup E[2]$  and  $E[< 0] = E[-1] \cup E[-2]$ . So there are fourteen ways we can express  $x$  in the form  $\lambda - a + b + z$ . We leave the details to the interested reader. For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 21\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 19$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$ . Hence  $x = \lambda + z$ . For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 19\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 17$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = \langle \lambda, \delta^\vee \rangle$ . Hence  $x = \lambda + z$ . For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 17\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 15$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by eight. Also,  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . As a result, there are numerous ways we can express  $x$  in the form  $\lambda - a + b + z$ . We leave the details to the interested reader. Further, for  $\alpha \in \Pi \setminus J$ , the bounds on  $\langle x, \alpha^\vee \rangle$  allow for the possibility that  $\langle \nu, \alpha^\vee \rangle \neq 0$ . By analyzing the constraints placed on  $a$ ,  $b$ , and  $z$  (to afford a nonzero  $\nu$ ), the resulting possibilities are all computed with MAGMA, and one finds that the only  $x$  that works is precisely  $-w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

$l = 13$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by three. Also  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . As in the type  $E_7$ ,  $l = 7$  case, there are ten ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 13\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 11$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by two. Also  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . As in the type  $E_7$ ,  $l = 9$  case, there are five ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$  one finds the bounds on  $\langle x, \alpha^\vee \rangle$  listed in Appendix A.2. In order to have  $x = -w_{0,J}(w \cdot 0) + 11\nu$ , we must have  $\langle \nu, \alpha^\vee \rangle = 0$ .

$l = 9$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by three. Also,  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . As in the type  $E_7$ ,  $l = 7$  case, there are ten ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$ , the bounds on  $\langle x, \alpha^\vee \rangle$  allow for the possibility that  $\nu \neq 0$ . To reduce the number of possibilities that need to be checked, observe that there is a sizable difference between  $\langle \lambda, \alpha_1^\vee \rangle = 18$  and  $\langle -w_{0,J}(w \cdot 0), \alpha_1^\vee \rangle = 8$ . Since  $\langle \nu, \alpha_1^\vee \rangle = 0$ , we must have  $\langle -a + b + z, \alpha_1^\vee \rangle = -10$ . We find that  $\langle -a + b, \alpha_1^\vee \rangle \geq -3$  and  $\langle z, \alpha_1^\vee \rangle \geq -8$ . This reduces the possibilities to a number manageable for MAGMA to compute all the possible cases. The only  $x$  that works is precisely  $-w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

$l = 7$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by 8 which is larger than 7. Here we may not immediately conclude that  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in J$ . However, since  $\langle \nu, \alpha^\vee \rangle \geq 0$  for all  $\alpha \in J$ , this must be true for all but possibly one  $\alpha$  for which one could have  $\langle \nu, \alpha^\vee \rangle = 1$ .

Suppose the latter case holds. Then we would have  $\langle x, \delta^\vee \rangle = 49$  whereas  $\langle \lambda, \delta^\vee \rangle = 50$ . So there would be only two ways in which  $x$  could occur ( $x = \lambda - x_1 + z$  where  $x_1 \in E[1]$  or  $x = \lambda + y_1 + z$  where  $y_1 \in E[-1]$ ). Here  $\Pi \setminus J = \{\alpha_4\}$  and one finds that  $-21 \leq \langle x, \alpha_4^\vee \rangle \leq -19$ . On the other

hand,  $\langle -w_{0,J}(w \cdot 0), \alpha_4^\vee \rangle = -17$ . Since  $\langle \nu, \alpha_4^\vee \rangle$  is an integer, these bounds show that we cannot have  $x = -w_{0,J}(w \cdot 0) + 7\nu$  for any  $\nu$ , contradicting our assumption. Therefore,  $\langle \nu, \alpha^\vee \rangle = 0$  for all  $\alpha \in J$ .

Now, our standard argument can be used to show that  $\langle \nu, \alpha_4^\vee \rangle = 0$ . Here  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . Since  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by 8 there are numerous ways we can express  $x$  in the form  $\lambda - a + b + z$ . One finds that  $-21 \leq \langle x, \alpha_4^\vee \rangle \leq -12$ . Since  $\langle -w_{0,J}(w \cdot 0), \alpha_4^\vee \rangle = -17$ , In order to have  $x = -w_{0,J}(w \cdot 0) + 7\nu$ , we must have  $\langle \nu, \alpha_4^\vee \rangle = 0$ .

Type  $F_4$ :

$l = 11$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . However, for  $\alpha \in \Pi \setminus J$ , one finds that  $-4 \leq \langle z, \alpha^\vee \rangle \leq 5$ . Hence,  $z$  cannot equal  $11\nu$  unless  $\nu = 0$ .

$l = 9$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by two. Also  $E[> 0] = E[1] \cup E[2]$  and  $E[< 0] = E[-1] \cup E[-2]$ . As in the type  $E_7$ ,  $l = 9$  case, there are five ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$ , the bounds on  $\langle x, \alpha^\vee \rangle$  allow for the possibility that  $\langle \nu, \alpha^\vee \rangle \neq 0$ . Using MAGMA, we compute all possibilities and see that the only  $x$  that works is precisely  $-w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

$l = 7$ : In this case,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$  and  $\langle \lambda, \delta^\vee \rangle$  differ by one. Also,  $E[> 0] = E[1] \cup E[2] \cup E[3]$  and  $E[< 0] = E[-1] \cup E[-2] \cup E[-3]$ . If we express  $x$  in the form  $\lambda - a + b + z$ , then  $a$  can contain terms only from  $E[1]$  and  $b$  can contain terms only from  $E[-1]$ . As in the type  $E_6$ ,  $l = 5$  case, there are two ways we can express  $x$  in the form  $\lambda - a + b + z$ . For  $\alpha \in \Pi \setminus J$ , the bounds on  $\langle x, \alpha^\vee \rangle$  allow for the possibility that  $\langle \nu, \alpha^\vee \rangle \neq 0$ . Using MAGMA, we compute all possibilities and see that the only  $x$  that works is precisely  $-w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

$l = 5$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence,  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . Here  $\Pi \setminus J = \{\alpha_2\}$ , and one finds that  $0 \leq \langle z, \alpha_2^\vee \rangle \leq 1$ . Hence  $z$  cannot equal  $5\nu$  unless  $\nu = 0$ .

Type  $G_2$ :

$l = 5$ : Here  $\lambda = -w_{0,J}(w \cdot 0)$ . Hence  $x = \lambda + z$ , and  $z$  would need to equal  $l\nu$ . Here  $E[0] = \{3\alpha_1 + 2\alpha_2 = \varpi_2\}$ . Hence, the only non-empty option for  $z$  is  $z = \varpi_2$  which is not equal to  $5\nu$  for any  $\nu$ . Hence,  $x = -w_{0,J}(w \cdot 0)$ , i.e.,  $\nu = 0$ .

## CHAPTER 5

### The Cohomology Algebra $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$

The identification of  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  for small  $l$  will proceed in several steps. The computation is motivated by the analogous problem for the restricted Lie algebra  $\mathfrak{g}_F$  over the algebraically closed field  $F$  of positive characteristic  $p$ . In that case, the support variety  $\mathcal{V}_{\mathfrak{g}_F}(F)$  of the trivial module  $F$  is homeomorphic (as a topological space) to the restricted nullcone  $\mathcal{N}_1(\mathfrak{g}_F) = \{x \in \mathfrak{g}_F \mid x^{[p]} = 0\}$ . By [CLNP], the variety  $\mathcal{N}_1(\mathfrak{g}_F)$  identifies with the closed subset  $G \cdot \mathfrak{u}_J \subset \mathfrak{g}_F$ , for an appropriate subset  $J \subset \Pi$ . When  $p \geq h$ ,  $J = \emptyset$  and  $\mathfrak{u}_J = \mathfrak{u}$ .

To attack the computation of the cohomology of  $u_\zeta(\mathfrak{g})$ , we consider the parabolic subgroup  $P_J$  associated to this subset  $J \subset \Pi$  with  $w(\Phi_0^+) = \Phi_{w \cdot 0}^+ = \Phi_J^+$ . Then we proceed as follows:

- In Section 5.1, the cohomology of  $u_\zeta(\mathfrak{g})$  is shown to be related to that of  $u_\zeta(\mathfrak{p}_J)$ .
- In Section 5.2, the cohomology of  $u_\zeta(\mathfrak{p}_J)$  is shown to be related to the cohomology of  $u_\zeta(\mathfrak{u}_J)$ .
- Sections 5.3–5.4 present the key computation for  $H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ .
- Sections 5.5–5.7 complete the computation of  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ .

Throughout this chapter, fix  $l$  satisfying Assumption 1.2.1. Fix  $w \in W$  and  $J \subseteq \Pi$  so that  $w(\Phi_0^+) = \Phi_J^+$ .

While the goal of this paper is to make cohomological computations in the case that  $l < h$ , the arguments are also valid for  $l \geq h$ . In that case, we would have  $\Phi_0 = \emptyset$ ,  $J = \emptyset$ , and  $w = \text{Id}$ . Then  $P_J = B$ ,  $U_J = U$ ,  $L_J = T$ ,  $\mathfrak{p}_J = \mathfrak{b}$ ,  $\mathfrak{u}_J = \mathfrak{u}$ ,  $\mathfrak{l}_J = \mathfrak{t}$ , and the module  $M = (\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* = \mathbb{C}$ . As such, our results recapture the calculation for  $l > h$  given by Ginzburg and Kumar in [GK].

#### 5.1. Spectral sequences, I

By Lemma 4.1.1,  $w \cdot 0$  is  $J$ -dominant, i.e.,  $\langle w \cdot 0, \alpha^\vee \rangle$  is a non-negative integer for all  $\alpha \in J$ . Using the factorization

$$\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} = \text{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} \circ \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)}$$

of functors, there is a Grothendieck spectral sequence

$$(5.1.1) \quad E_2^{i,j} = R^i \text{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} R^j \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \Rightarrow R^{i+j} \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w \cdot 0.$$

However, since  $w \cdot 0$  is  $J$ -dominant,  $R^j \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 = 0$  for  $j > 0$ . Consequently, this spectral sequence collapses and so yields, by [A, Cor. 3.8],

$$(5.1.2) \quad R^i \text{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})} \left( \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \right) = R^i \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w \cdot 0 = \begin{cases} \mathbb{C} & \text{if } i = \ell(w) \\ 0 & \text{if } i \neq \ell(w). \end{cases}$$

The following spectral sequence provides a connection between the cohomology of  $u_\zeta(\mathfrak{p}_J)$  and that of  $u_\zeta(\mathfrak{g})$ .

**Theorem 5.1.1.** *Let  $w \in W$  such that  $w(\Phi_0^+) = \Phi_J^+$  where  $J \subseteq \Pi$ . There exists a first quadrant spectral sequence of rational  $G$ -modules*

$$E_2^{i,j} = R^i \text{ind}_{P_J}^G H^j \left( u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \right) \Rightarrow H^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$



PROOF. We follow the construction in [Jan1, I 6.12]. Form the functors  $\mathcal{F}_1, \mathcal{F}_2 : U_\zeta(\mathfrak{p}_J)\text{-mod} \rightarrow G\text{-mod}$  defined by setting

$$\mathcal{F}_1(-) = \text{Hom}_{u_\zeta(\mathfrak{g})}(\mathbb{C}, \text{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})}(-)) \text{ and } \mathcal{F}_2(-) = \text{ind}_{P_J}^G \text{Hom}_{u_\zeta(\mathfrak{p}_J)}(\mathbb{C}, -).$$

The reader should observe that we are implicitly using the Frobenius map (cf. Section 2.3) and the following identification of functors:

$$\text{ind}_{P_J}^G(-) \cong \text{ind}_{U_\zeta(\mathfrak{p}_J)/u_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})/u_\zeta(\mathfrak{g})}(-).$$

The functors  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are naturally isomorphic. Therefore, there exist two Grothendieck spectral sequences:

$$(5.1.3) \quad \begin{cases} {}'E_2^{i,j} = H^i(u_\zeta(\mathfrak{g}), R^j \text{ind}_{U_\zeta(\mathfrak{p}_J)}^{U_\zeta(\mathfrak{g})}(\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)) \Rightarrow (R^{i+j} \mathcal{F}_1)(\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0); \\ E_2^{i,j} = R^i \text{ind}_{P_J}^G H^j(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)) \Rightarrow (R^{i+j} \mathcal{F}_2)(\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \end{cases}$$

necessarily converging to the same abutment.

By (5.1.2), the first spectral sequence  ${}'E_2^{i,j}$  collapses, leading to an identification

$$(R^{\bullet+\ell(w)} \mathcal{F}_1)(\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong H^\bullet(u_\zeta(\mathfrak{g}), R^{\ell(w)} \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})} w \cdot 0) \cong H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

Combining this with the second spectral sequence  $E_2^{i,j}$  proves the theorem.  $\square$

## 5.2. Spectral sequences, II

In this section, we reidentify the term  $H^j(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)$  occurring in the spectral sequence in Theorem 5.1.1. Using the Lyndon-Hochschild-Serre spectral sequence in Lemma 2.8.1 for  $u_\zeta(\mathfrak{u}_J) \trianglelefteq u_\zeta(\mathfrak{p}_J)$ . Note that  $u_\zeta(\mathfrak{p}_J)/u_\zeta(\mathfrak{u}_J) \cong u_\zeta(\mathfrak{l}_J)$ . Thus, there is a spectral sequence

$$E_2^{i,j} = H^i(u_\zeta(\mathfrak{l}_J), H^j(u_\zeta(\mathfrak{u}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0).$$

All of the cohomology groups involved admit an action of  $U_\zeta(\mathfrak{p}_J)$  induced from the  $\text{Ad}_r$ -action. Furthermore, the action of the subalgebra  $u_\zeta(\mathfrak{p}_J)$  is trivial. Hence, there is an action of  $\mathbb{U}(\mathfrak{p}_J)$  (or equivalently  $P_J$ ). Moreover, the spectral sequence preserves this action.

Since  $\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0$  is trivial as a  $U_\zeta(\mathfrak{u}_J)$ -module, the left-hand side of the spectral sequence may be reidentified as follows:

$$(5.2.1) \quad E_2^{i,j} = H^i(u_\zeta(\mathfrak{l}_J), H^j(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0).$$

**Proposition 5.2.1.** *Let  $w \in W$  and  $J \subset \Pi$  be such that  $w(\Phi_0^+) = \Phi_J^+$ . Then for all  $j \geq 0$  there is an isomorphism of  $\mathbb{U}(\mathfrak{p}_J)$ -modules*

$$H^j(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^j(u_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

PROOF. Since  $\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0$  is projective as a  $u_\zeta(\mathfrak{l}_J)$ -module, in the spectral sequence (5.2.1),  $E_2^{i,j} = 0$  for all  $i > 0$ . Thus the spectral sequence collapses giving for all  $j \geq 0$

$$\begin{aligned} H^j(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) &\cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(\mathbb{C}, H^j(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \\ &\cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^j(u_\zeta(\mathfrak{u}_J), \mathbb{C})). \end{aligned}$$

$\square$

### 5.3. An identification theorem

The following theorem now gives an identification as  $U_\zeta^0$ -modules (or equivalently  $\mathcal{U}(\mathfrak{h})$ -modules where  $\mathfrak{h} \subset \mathfrak{g}$  is the Cartan subalgebra) of the Hom-group appearing in Proposition 5.2.1.

**Theorem 5.3.1.** *Let  $l$  be as in Assumption 1.2.1 and  $w \in W$  such that  $w(\Phi_0^+) = \Phi_J^+$ .*

- (a) *Suppose that  $l \nmid n+1$  when  $\Phi$  is of type  $A_n$  and  $l \neq 9$  when  $\Phi$  is of type  $E_6$ . Then as  $U_\zeta^0$ -modules*

$$\mathrm{Hom}_{u_\zeta(l_J)} \left( (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^s(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \right) \cong S^{\frac{s-\ell(w)}{2}} (\mathfrak{u}_J^*)^{[1]}.$$

- (b) *If  $\Phi$  is of type  $A_n$  with  $n+1 = l(m+1)$  and  $w \in W$  is as defined in (4.8.1), then as  $U_\zeta^0$ -modules*

$$\mathrm{Hom}_{u_\zeta(l_J)} \left( (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^s(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \right) \cong \bigoplus_{t=0}^{l-1} S^{\frac{s-\ell(w)-(m+1)t(l-t)}{2}} (\mathfrak{u}_J^*)^{[1]} \otimes l\varpi_{t(m+1)},$$

where  $\varpi_0 = 0$ .

- (c) *If  $\Phi$  is of type  $E_6$  and  $l = 9$  (assuming that  $w$  and  $J$  are as in Appendix A.1), then as  $U_\zeta^0$ -modules*

$$\begin{aligned} \mathrm{Hom}_{u_\zeta(l_J)} \left( (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^s(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \right) &\cong S^{\frac{s-\ell(w)}{2}} (\mathfrak{u}_J^*)^{[1]} \oplus \left( S^{\frac{s-20}{2}} (\mathfrak{u}_J^*)^{[1]} \otimes l\varpi_1 \right) \\ &\oplus \left( S^{\frac{s-20}{2}} (\mathfrak{u}_J^*)^{[1]} \otimes l\varpi_6 \right). \end{aligned}$$

PROOF. For convenience set  $M = (\mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*$  and  $\mathcal{G}(-) = \mathrm{Hom}_{u_\zeta(l_J)}(M, -)$ . Since the module  $M$  is injective (equivalently projective) as a  $u_\zeta(l_J)$ -module, the functor  $\mathcal{G}(-) = \mathrm{Hom}_{u_\zeta(l_J)}(M, -)$  is an exact functor.

The argument will proceed by induction on successive quotients of  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  (cf. [GK, 2.4]). Let  $N = |\Phi^+ \setminus \Phi_J^+|$ . Note that previously  $N$  was used to denote  $|\Phi^+|$ , but this should not cause any confusion here. As in Chapter 2, choose any fixed ordering of root vectors  $f_1, f_2, \dots, f_N$  in  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  corresponding to the positive roots in  $\Phi^+ \setminus \Phi_J^+$ . For purposes of this argument, the precise ordering is irrelevant. Each  $f_i^l$  is central in  $\mathcal{U}_\zeta(\mathfrak{u}_J)$ . For  $0 \leq i \leq N$ , set  $A_i = \mathcal{U}_\zeta(\mathfrak{u}_J) / \langle f_1^l, f_2^l, \dots, f_i^l \rangle$  where  $\langle \dots \rangle$  denotes “the subalgebra generated by  $\dots$ ” (with  $A_0 = \mathcal{U}_\zeta(\mathfrak{u}_J)$ ). Note that  $A_N = u_\zeta(\mathfrak{u}_J)$ . For  $1 \leq i \leq N$ , set  $B_i = \langle f_i^l \rangle \subset A_{i-1}$  be the subalgebra generated by  $f_i^l$ . Note that each  $B_i$  is a polynomial algebra in one variable. Note also that  $B_i$  is normal (in fact central) in  $A_{i-1}$ , and  $A_{i-1}/B_i \cong A_i$ . For  $0 \leq i \leq N$ , let  $V_i$  be an  $i$ -dimensional vector space with basis  $\{x_1, x_2, \dots, x_i\}$ . Further consider  $V_i$  as a  $U_\zeta^0$ -module by letting  $x_i$  have weight  $\gamma_i$ . That is, each  $x_i$  is “dual” to the element  $f_i$ . In particular,  $V_N \cong \mathfrak{u}_J^*$ . Here  $V_0 = \{0\}$ .

Consider first part (a). We prove inductively for  $0 \leq i \leq N$  that as  $U_\zeta^0$ -modules we have

$$\mathcal{G}(H^s(A_i, \mathbb{C})) \cong \begin{cases} S^r(V_i)^{[1]} & \text{if } s = 2r + \ell(w) \\ 0 & \text{else.} \end{cases}$$

The case  $i = N$  is precisely the statement of the theorem. For  $i = 0$ , this is precisely Theorem 4.3.1 where by convention we take  $S^0(V_0) = \mathbb{C}$ .

Assume now that the claim is true for  $i-1$ , and we will show that it is true for  $i$ . We will make use of the LHS spectral sequence of Lemma 2.8.1 for  $B_i \trianglelefteq A_{i-1}$ . Since  $B_i$  is a polynomial algebra in one variable, its cohomology is an exterior algebra in one variable. Precisely, for each  $i$ , we have as a

$U_\zeta^0$ -module:

$$H^b(B_i, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } b = 0 \\ \mathbb{C}_i^{[1]} & \text{if } b = 1 \\ 0 & \text{else,} \end{cases}$$

where  $\mathbb{C}_i$  is a one-dimensional vector space with basis element  $x_i$  (i. e., of weight  $\gamma_i$ ).

The spectral sequence is

$$E_2^{a,b} = H^a(A_i, H^b(B_i, \mathbb{C})) \Rightarrow H^{a+b}(A_{i-1}, \mathbb{C}).$$

The algebra  $A_i$  acts on  $B_i$  via the  $\overline{\text{Ad}}$ -action. By Corollary 2.7.4(b), this action is trivial. Therefore this spectral sequence can be rewritten as

$$E_2^{a,b} = H^a(A_i, \mathbb{C}) \otimes H^b(B_i, \mathbb{C}) \Rightarrow H^{a+b}(A_{i-1}, \mathbb{C}).$$

Alternately, this easily follows from the above description of  $H^b(B_i, \mathbb{C})$ .

Again using the  $\overline{\text{Ad}}$ -action,  $u_\zeta(\mathfrak{l}_J)$  acts on all terms of the spectral sequence and this action is preserved by the differentials. Since the functor  $\mathcal{G}(-)$  is exact, we can apply it to the above spectral sequence and obtain a new spectral sequence. For convenience, we abusively use the same name:

$$E_2^{a,b} = \mathcal{G}(H^a(A_i, \mathbb{C}) \otimes H^b(B_i, \mathbb{C})) \Rightarrow \mathcal{G}(H^{a+b}(A_{i-1}, \mathbb{C})).$$

The above description of  $H^b(B_i, \mathbb{C})$  shows that  $u_\zeta(\mathfrak{l}_J)$  acts trivially on it. And hence, this spectral sequence may be rewritten as

$$E_2^{a,b} = \mathcal{G}(H^a(A_i, \mathbb{C})) \otimes H^b(B_i, \mathbb{C}) \Rightarrow \mathcal{G}(H^{a+b}(A_{i-1}, \mathbb{C})).$$

Observe that  $E_2^{a,b} = 0$  for  $b \geq 2$ . That is, the spectral sequence consists of at most two nonzero rows. So only the first differential  $d_2 : E_2^{a,1} \rightarrow E_2^{a+2,0}$  could potentially be nonzero. The first row  $E_2^{a,0} = H^a(A_i, \mathbb{C})$  is precisely what we are trying to identify inductively. Note also that for all  $a$ , the second row  $E_2^{a,1} \cong E_2^{a,0} \otimes \mathbb{C}_i^{[1]}$ . In particular,  $E_2^{a,1} \neq 0$  if and only if  $E_2^{a,0} \neq 0$ .

By the inductive hypothesis, we know that the abutment

$$\mathcal{G}(H^{a+b}(A_{i-1}, \mathbb{C})) \cong \begin{cases} S^r(V_{i-1})^{[1]} & \text{if } a+b = 2r + \ell(w) \\ 0 & \text{else.} \end{cases}$$

In particular,  $\mathcal{G}(H^{a+b}(A_{i-1}, \mathbb{C})) = 0$  for  $a+b < \ell(w)$ .

Let  $A \geq 0$  be least value of  $a$  such that  $E_2^{a,0} \neq 0$ . Hence,  $A$  is necessarily the least value of  $a$  such that  $E_2^{a,1} \neq 0$ . In particular,  $E_2^{A-2,1} = 0$  and hence  $E_\infty^{A,0} \cong E_2^{A,0}/d_2(E_2^{A-2,1}) = E_2^{A,0}$ . By the inductive hypothesis, we conclude that  $A = \ell(w)$ . So for all  $a < \ell(w)$  we have

$$\mathcal{G}(H^a(A_i, \mathbb{C})) = 0.$$

Next we claim that  $E_2^{\ell(w)+a,0} = 0 = E_2^{\ell(w)+a,1}$  for all odd  $a > 0$ . This can be seen inductively on  $a$ . For example, since  $E_2^{\ell(w)-1,1} = 0$ ,

$$E_2^{\ell(w)+1,0} = E_2^{\ell(w)+1,0}/d_2(E_2^{\ell(w)-1,1}) \cong E_\infty^{\ell(w)+1,0} \subset \mathcal{G}(H^{\ell(w)+1}(A_{i-1}, \mathbb{C})) = 0.$$

Inductively, for odd  $a > 0$ , we similarly have  $E_2^{\ell(w)+a-2,1} = 0$  and so

$$E_2^{\ell(w)+a,0} = E_2^{\ell(w)+a,0}/d_2(E_2^{\ell(w)+a-2,1}) \cong E_\infty^{\ell(w)+a,0} \subset \mathcal{G}(H^{\ell(w)+a}(A_{i-1}, \mathbb{C})) = 0.$$

In other words, for all odd  $a > 0$ , we have (as claimed)

$$\mathcal{G}(H^{\ell(w)+a}(A_i, \mathbb{C})) = 0.$$

Summarizing: our spectral sequence has  $E_2^{a,b} = 0$  for  $a < \ell(w)$  and  $E_2^{\ell(w)+a,b} = 0$  for all odd  $a > 0$ . That is, the columns are initially zero and then begin to alternate nonzero (potentially) and zero thereafter.

Furthermore, for all even  $a \geq 0$ , we then have

$$\ker\{d_2 : E_2^{\ell(w)+a,1} \rightarrow E_2^{\ell(w)+a+2,0}\} \subset E_\infty^{\ell(w)+a,1} \subset \mathcal{G}(\mathbf{H}^{\ell(w)+a+1}(A_{i-1}, \mathbb{C})) = 0.$$

Therefore  $d_2 : E_2^{\ell(w)+a,1} \rightarrow E_2^{\ell(w)+a+2,0}$  is injective. Hence, we have for all even  $a \geq 0$ ,

$$\begin{aligned} E_2^{\ell(w)+a,0} / E_2^{\ell(w)+a-2,1} &\cong E_2^{\ell(w)+a,0} / d_2(E_2^{\ell(w)+a-2,1}) \cong E_\infty^{\ell(w)+a,0} \\ &\cong \mathcal{G}(\mathbf{H}^{\ell(w)+a}(A_{i-1}, \mathbb{C})) = S^{a/2}(V_{i-1})^{[1]}. \end{aligned}$$

So we have a short exact sequence of  $U_\zeta^0$ -modules:

$$0 \rightarrow E_2^{\ell(w)+a-2,1} \rightarrow E_2^{\ell(w)+a,0} \rightarrow S^{a/2}(V_{i-1})^{[1]} \rightarrow 0.$$

But identifying these  $E_2$ -terms gives

$$0 \rightarrow \mathcal{G}(\mathbf{H}^{\ell(w)+a-2}(A_i, \mathbb{C})) \otimes \mathbb{C}_i^{[1]} \rightarrow \mathcal{G}(\mathbf{H}^{\ell(w)+a}(A_i, \mathbb{C})) \rightarrow S^{a/2}(V_{i-1})^{[1]} \rightarrow 0.$$

Inducting now on even  $a \geq 0$ , we may assume that

$$\mathcal{G}(\mathbf{H}^{\ell(w)+a-2}(A_i, \mathbb{C})) = S^{(a-2)/2}(V_i)^{[1]}.$$

The short exact sequence becomes

$$0 \rightarrow S^{(a-2)/2}(V_i)^{[1]} \otimes \mathbb{C}_i^{[1]} \rightarrow \mathcal{G}(\mathbf{H}^{\ell(w)+a}(A_i, \mathbb{C})) \rightarrow S^{a/2}(V_{i-1})^{[1]} \rightarrow 0$$

or setting  $a = 2r$ ,

$$0 \rightarrow S^{r-1}(V_i)^{[1]} \otimes \mathbb{C}_i^{[1]} \rightarrow \mathcal{G}(\mathbf{H}^{\ell(w)+2r}(A_i, \mathbb{C})) \rightarrow S^r(V_{i-1})^{[1]} \rightarrow 0.$$

Hence as a  $U_\zeta^0$ -module,

$$\mathcal{G}(\mathbf{H}^{\ell(w)+2r}(A_i, \mathbb{C})) \cong (S^{r-1}(V_i)^{[1]} \otimes \mathbb{C}_i^{[1]}) \oplus S^r(V_{i-1})^{[1]}.$$

The left hand factor consists of  $r$ -fold symmetric powers in the  $x_j$  which contain at least one  $x_i$ , while the righthand factor consists of  $r$ -fold symmetric powers in  $x_j$  with  $1 \leq j \leq i-1$ . Hence

$$\mathcal{G}(\mathbf{H}^{\ell(w)+2r}(A_i, \mathbb{C})) \cong S^r(V_i)^{[1]}$$

which along with the above conclusions verifies the inductive claim and hence part (a) of the theorem.

For parts (b) and (c) a similar argument can be used whose details are left to the interested reader. Of crucial importance here is the degrees in which the “extra” classes arise in parts (b) and (c) of Theorem 4.3.1. For example, consider part (b). We have  $\mathcal{G}(\mathbf{H}^i(\mathcal{U}_\zeta(\mathbf{u}_J), \mathbb{C})) = 0$  unless  $i = \ell(w) + (m+1)t(l-t)$  for  $0 \leq t \leq l-1$ . Observe that  $t(l-t)$  (and, hence,  $(m+1)t(l-t)$ ) is necessarily even. Thus the extra cohomology classes appear in degrees having the same parity as  $\ell(w)$ . As such, in the above argument, we will have a similar phenomenon happening in the spectral sequence:  $E_2^{a,b} = 0$  for  $a < \ell(w)$  or  $a = \ell(w) + a'$  with  $a' > 0$  being odd, and the analogous argument will give the claim. Similarly in part (c), the extra cohomology classes lie in a degree with the same parity as  $\ell(w)$ .

□

#### 5.4. Spectral sequences, III

As mentioned previously, the Hom-groups in Theorem 5.3.1 admit an action of  $\mathbb{U}(\mathfrak{p}_J)$  induced from the  $\text{Ad}_r$ -action. On the other hand  $\mathbb{U}(\mathfrak{p}_J)$  acts naturally on  $\mathfrak{u}_J$  by the adjoint action or on  $\mathfrak{u}_J^*$  by the coadjoint action. This can be further extended to an action on  $S^\bullet(\mathfrak{u}_J^*)$ . With this action, the isomorphisms in Theorem 5.3.1 also hold as  $\mathbb{U}(\mathfrak{p}_J)$ -modules (or equivalently  $P_J$ -modules).

**Lemma 5.4.1.** *The isomorphisms of Theorem 5.3.1 also hold as  $\mathbb{U}(\mathfrak{p}_J)$ -modules where the actions are as described above.*

PROOF. Consider first the generic case - part (a). We use notation as in the proof of Theorem 5.3.1. Let  $Z_J = \langle f_1^l, \dots, f_N^l \rangle \subset \mathcal{U}_\zeta(\mathfrak{u}_J)$  be the central subalgebra such that  $\mathcal{U}_\zeta(\mathfrak{u}_J)/Z_J \cong u_\zeta(\mathfrak{u}_J)$  (cf. also Section 2.7). Consider the spectral sequence

$$E_2^{a,b} = H^a(u_\zeta(\mathfrak{u}_J), H^b(Z_J, \mathbb{C})) \Rightarrow H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$$

given by Lemma 2.8.1. This is a spectral sequence of  $U_\zeta(\mathfrak{p}_J)$ -modules. Since  $u_\zeta(\mathfrak{p}_J)$  acts trivially on  $H^b(Z_J, \mathbb{C})$  (cf. Corollary 2.7.4(b)), this may be rewritten as

$$E_2^{a,b} = H^a(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes H^b(Z_J, \mathbb{C}) \Rightarrow H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C}).$$

Furthermore, applying the functor  $\mathcal{G}(-)$  we get a new spectral sequence (using the same name)

$$E_2^{a,b} = \mathcal{G}(H^a(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \otimes H^b(Z_J, \mathbb{C})) \Rightarrow \mathcal{G}(H^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})),$$

whose differentials still preserve the action of  $U_\zeta(\mathfrak{p}_J)$ . Since  $u_\zeta(\mathfrak{u}_J)$  acts trivially on both  $H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$  and  $H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ , and  $u_\zeta(\mathfrak{l}_J) \cong u_\zeta(\mathfrak{p}_J)/u_\zeta(\mathfrak{u}_J)$ , we have

$$\mathcal{G}(H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})) = \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong \text{Hom}_{u_\zeta(\mathfrak{p}_J)}(M, H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}))$$

and

$$\mathcal{G}(H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})) = \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong \text{Hom}_{u_\zeta(\mathfrak{p}_J)}(M, H^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

Therefore,  $u_\zeta(\mathfrak{p}_J)$  acts trivially on this new spectral sequence, and so this is a spectral sequence of  $\mathbb{U}(\mathfrak{p}_J)$ -modules.

By Theorem 4.3.1, the abutment is nonzero only when  $a + b = \ell(w)$  in which case it is the trivial module  $\mathbb{C}$ . And by Theorem 5.3.1, as  $\mathbb{U}(\mathfrak{h})$ -modules, we know that  $\mathcal{G}(H^a(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong S^{\frac{a-\ell(w)}{2}}(\mathfrak{u}_J^*)$ . Also, the algebra  $Z_J$  is a polynomial algebra and so its cohomology as an algebra is an exterior algebra. Moreover, by [ABG, Cor. 2.9.6], as  $\mathbb{U}(\mathfrak{h})$ -modules,  $H^\bullet(Z_J, \mathbb{C}) \cong \Lambda^\bullet(\mathfrak{u}_J^*)$  (i.e., the ordinary exterior algebra on  $\mathfrak{u}_J^*$ ), where the action of  $\mathbb{U}(\mathfrak{h})$  on  $\Lambda^\bullet(\mathfrak{u}_J^*)$  is given by the coadjoint action. Since  $H^\bullet(Z_J, \mathbb{C})$  is a  $\mathbb{U}(\mathfrak{p}_J)$ -module and the coadjoint action on  $\Lambda^\bullet(\mathfrak{u}_J^*)$  can be extended to  $\mathfrak{p}_J$ , by applying  $\text{ind}_{\mathbb{U}(\mathfrak{h})}^{\mathbb{U}(\mathfrak{p}_J)}(-)$ ,  $H^\bullet(Z_J, \mathbb{C}) \cong \Lambda^\bullet(\mathfrak{u}_J^*)$  as  $\mathfrak{p}_J$ -modules.

As in Theorem 5.3.1,  $E_2^{a,b} = 0$  for  $a < \ell(w)$  and  $E_2^{\ell(w)+a,b} = 0$  for odd  $a$ . Consider the term

$$E_2^{\ell(w)+2,0} = \mathcal{G}(H^{\ell(w)+2}(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong S^1(\mathfrak{u}_J^*) = \mathfrak{u}_J^*,$$

where the latter identifications are as  $\mathbb{U}(\mathfrak{h})$ -modules. We will see this also holds as  $\mathbb{U}(\mathfrak{p}_J)$ -modules. Since  $E_\infty^{\ell(w),1} = 0$  and  $E_\infty^{\ell(w)+2,0} = 0$ , the differential  $d_2 : E_2^{\ell(w),1} \rightarrow E_2^{\ell(w)+2,0}$  is an isomorphism of  $\mathbb{U}(\mathfrak{p}_J)$ -modules.

Since  $E_2^{\ell(w),1} = H^1(Z_J, \mathbb{C}) \cong \mathfrak{u}_J^*$  as a  $\mathbb{U}(\mathfrak{p}_J)$ -module and  $E_2^{\ell(w)+2,0} = \mathcal{G}(H^{\ell(w)+2}(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong \mathfrak{u}_J^*$  as a  $\mathbb{U}(\mathfrak{h})$ -module, this latter identification must also hold as a  $\mathbb{U}(\mathfrak{p}_J)$ -module.

As already noted, the  $\mathbb{U}(\mathfrak{p}_J)$ -structure of the terms

$$E_2^{\ell(w),b} = H^b(Z_J, \mathbb{C}) \cong \Lambda^b(\mathfrak{u}_J^*)$$

is determined by the coadjoint action. Moreover, we can now say that the  $\mathbb{U}(\mathfrak{p}_J)$ -structure of the terms

$$E_2^{\ell(w)+2,b} = \mathcal{G}(H^{\ell(w)+2}(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \otimes H^b(Z_J, \mathbb{C}) \cong \mathfrak{u}_J^* \otimes \Lambda^b(\mathfrak{u}_J^*)$$

is determined by the coadjoint action. Therefore, since  $E_\infty^{\ell(w)+4,0} = 0$ , the  $\mathbb{U}(\mathfrak{p}_J)$ -structure of

$$E_2^{\ell(w)+4,0} = \mathcal{G}(H^{\ell(w)+4}(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong S^2(\mathfrak{u}_J^*)$$

(with isomorphism as a  $\mathbb{U}(\mathfrak{h})$ -module), which is determined by the structure on  $E_2^{\ell(w),b}$  and  $E_2^{\ell(w)+2,b}$ , must also be determined by the coadjoint action. Inductively, we conclude that indeed the  $\mathbb{U}(\mathfrak{p}_J)$ -action on

$$\mathcal{G}(H^a(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \cong S^{\frac{a-\ell(w)}{2}}(\mathfrak{u}_J^*)$$

is given by the coadjoint action.

For parts (b) and (c), a similar argument may be used. The key here (as in Theorem 5.3.1) is that the additional classes in  $\mathcal{G}(H^i(\mathfrak{u}_J, \mathbb{C}))$  lie in degrees which have the same parity as  $\ell(w)$ . Furthermore, the additional classes have distinct (nonzero) weights whose differences are neither sums of positive root nor sums of negative roots as noted in Remark 4.3.2.  $\square$

### 5.5. Proof of main result, Theorem 1.2.3, I

In this section, we present a proof of the isomorphisms in Theorem 1.2.3 as  $G$ -modules. Let us recall the following vanishing result due to Broer [Br2, Thm. 2.2] and extended to a more general setting by Sommers [So1, Prop. 4]. Let  $J$  be an arbitrary subset of  $\Pi$ . Then

$$(5.5.1) \quad R^i \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \otimes \lambda = 0$$

for  $i > 0$  and  $\lambda$  is a  $P$ -regular weights [KLT, Section 1.1] inside the character group  $X(P_J)$ . This vanishing result is proved using the Grauert-Riemenschneider theorem. We can now prove the first of our main theorems which provides a precise description of the cohomology of quantum groups at roots of unity in the case that  $\zeta$  is a primitive  $l$ th root of unity.

PROOF. Consider the cases listed in (b)(i). According to Proposition 5.2.1, Theorem 5.3.1(a), and Lemma 5.4.1, we have, as  $\mathbb{U}(\mathfrak{p}_J)$ -modules,

$$H^j(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{h})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0) \cong S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}_J^*).$$

By substituting this identification into the spectral sequence given in Theorem 5.1.1, we have

$$E_2^{i,j} = R^i \text{ind}_{P_J}^G S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}_J^*) \Rightarrow H^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

We can now apply (5.5.1) to conclude that  $E_2^{i,j} = 0$  for  $i > 0$ , thus the spectral sequence collapses to yield

$$H^s(u_\zeta(\mathfrak{g}), \mathbb{C}) = \text{ind}_{P_J}^G S^{\frac{s}{2}}(\mathfrak{u}_J^*).$$

This gives part (a) for those types listed in (b)(i). According to Theorem 3.6.2,  $\text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \cdot \mathfrak{u}_J] \cong \mathbb{C}[\mathcal{N}(\Phi_0)]$ , which gives part (b)(i).

Consider now the cases listed in part (b)(ii). In this case we have the following spectral sequence (using Proposition 5.2.1, Theorem 5.3.1(b), and Lemma 5.4.1),

$$E_2^{i,j} = \bigoplus_{t=0}^{l-1} R^i \text{ind}_{P_J}^G S^{\frac{j-\ell(w)-(m+1)t(l-t)}{2}}(\mathfrak{u}_J^*) \otimes \varpi_{t(m-1)} \Rightarrow H^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

One can again apply (5.5.1) because we are tensoring the symmetric algebra by weights in  $X(P_J)_+$ , thus the spectral sequence collapses and yields part (ii) of (a) and (b). One can argue similarly for those cases listed in part (b)(iii) using Theorem 5.3.1(c). In that case  $\ell(w) = 8$ .  $\square$

### 5.6. Spectral sequences, IV

To show the isomorphisms in Theorem 1.2.3 are, in fact, isomorphisms of algebras, we will need some further observations on one of the spectral sequences introduced earlier. These observations will also be used in Section 6.3.

In Section 2.9, a filtration was introduced on  $\mathcal{U}_\zeta(\mathfrak{u}_J)$  which can be restricted to  $u_\zeta(\mathfrak{u}_J)$ . Since  $u_\zeta(\mathfrak{u}_J)$  is finite dimensional, the induced filtration on the cobar complex computing the cohomology  $H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$  is finite. As in the proof of part (b) of Proposition 2.9.1, there is a spectral sequence as follows.

**Lemma 5.6.1.** *There is a spectral sequence*

$$E_1^{i,j} = H^{i+j}(\text{gr } u_\zeta(\mathfrak{u}_J), \mathbb{C})_{(i)} \Rightarrow H^{i+j}(u_\zeta(\mathfrak{u}_J), \mathbb{C})$$

*of graded algebras and  $U_\zeta^0$ -modules.*

Since the filtration on  $u_\zeta(\mathfrak{u}_J)$  is finite, this spectral sequence has only finitely many columns, and hence eventually stops.

Globally, we have  $E_1^{\bullet,\bullet} \cong H^\bullet(\text{gr } u_\zeta(\mathfrak{u}_J), \mathbb{C})$ . By [GK, Prop. 2.3.1], there exists a graded algebra isomorphism

$$(5.6.1) \quad H^\bullet(\text{gr } u_\zeta(\mathfrak{u}_J), \mathbb{C}) \cong S^\bullet(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^\bullet.$$

This is also an isomorphism of  $U_\zeta^0$ -modules with  $u_\zeta^0$  acting trivially on the symmetric algebra. Moreover, under the isomorphism (5.6.1),

$$(5.6.2) \quad H^n(\text{gr } u_\zeta(\mathfrak{u}_J), \mathbb{C}) \cong \bigoplus_{2a+b=n} S^a(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^b.$$

By the isomorphism (5.6.1),  $E_1^{\bullet,\bullet}$  is finitely generated over a subalgebra which is isomorphic (as algebras and  $U_\zeta^0$ -modules) to  $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ . For notational convenience, we abusively consider  $S^\bullet(\mathfrak{u}_J^*)^{[1]}$  as a subalgebra of  $E_1^{\bullet,\bullet}$ . Under mild conditions on  $l$ , this subalgebra consists of universal cycles.

**Proposition 5.6.2.** *Let  $l$  satisfy Assumption 1.2.2 and  $J \subseteq \Pi$ . In the spectral sequence of Lemma 5.6.1,  $d_r(S^\bullet(\mathfrak{u}_J^*)^{[1]}) = 0$  for  $r \geq 1$ .*

PROOF. We first consider the case when  $r = 1$ . From (5.6.2), the submodule  $S^1(\mathfrak{u}_J^*)^{[1]}$  is identified with a submodule of  $H^2(\text{gr } u_\zeta(\mathfrak{u}_J), \mathbb{C})$ . As such, the image of  $S^1(\mathfrak{u}_J^*)^{[1]}$  under  $d_1$  must lie in

$$H^3(\text{gr } u_\zeta(\mathfrak{u}_J), \mathbb{C}) \cong (S^1(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^1) \oplus \Lambda_{\zeta,J}^3.$$

A  $U_\zeta^0$ -homogeneous element  $x_\sigma$  of  $S^1(\mathfrak{u}_J^*)^{[1]}$  has weight  $l\sigma$  for some  $\sigma \in \Phi^+ \setminus \Phi_J^+$ . Hence, by weight considerations, if the image of  $x_\sigma$  is not zero, it cannot lie in  $S^1(\mathfrak{u}_J^*)^{[1]} \otimes \Lambda_{\zeta,J}^1$ . On the other hand, for  $x_\sigma$  to have nonzero image in  $\Lambda_{\zeta,k}^3$ , we must have  $l\sigma = \gamma_1 + \gamma_2 + \gamma_3$  for three distinct (positive) roots  $\gamma_i \in \Phi^+ \setminus \Phi_J^+$ . Under the given conditions on  $l$ , this is not possible. To see this, we argue by the type of  $\Phi$ .

For any weight  $\eta$  which lies in the positive root lattice, we can write  $\eta = \sum_{\beta \in \Pi} n_{\eta,\beta} \beta$  for unique  $n_{\eta,\beta} \in \mathbb{Z}_{\geq 0}$ . Set  $\gamma := \gamma_1 + \gamma_2 + \gamma_3$  for  $\gamma_i$  as above. For type  $A_2$  there is only one element in  $\Lambda_{\zeta,k}^3$  and it is not of the form  $l\sigma$ . If the root system is not of type  $A_2$  the index of the root lattice in the weight lattice is not divisible by 3. Therefore, in order to have  $\gamma = l\sigma$ ,  $l$  must divide  $n_{\gamma,\beta}$  for each  $\beta \in \Pi$ .

In type  $A_n$ , for each  $\gamma_i$  and  $\beta \in \Pi$ , we have  $n_{\gamma_i,\beta} \leq 1$ . Hence,  $n_{\gamma,\beta} \leq 3$ , and the claim immediately follows for  $l > 3$ . For  $l = 3$ , we could only have  $3\sigma = \gamma$  if  $\gamma_1 = \gamma_2 = \gamma_3$  which contradicts our assumption.

For types  $B_n$ ,  $C_n$ , and  $D_n$ ,  $n_{\gamma,\beta} \leq 6$ . Hence, the claim follows if  $l \geq 7$ . Suppose  $l = 5$  and  $n_{\gamma,\beta} = 5$  for some  $\beta \in \Pi$ . Without a loss of generality we may assume that  $n_{\gamma_1,\beta} = 2$ ,  $n_{\gamma_2,\beta} = 2$ ,

and  $n_{\gamma_3, \beta} = 1$ . Observe that there is necessarily some  $\beta' \in \Pi$  such that  $n_{\gamma_1, \beta'} = 1$ ,  $n_{\gamma_2, \beta'} = 1$ , and  $n_{\gamma_3, \beta'} \in \{0, 1\}$ . Thus  $n_{\gamma, \beta'} \in \{2, 3\}$  and so cannot be divisible by 5. In types  $B_n$  and  $C_n$  when  $l = 3$ , the condition can be satisfied (e.g., in type  $B_2$ ,  $\alpha_1 + (\alpha_1 + \alpha_2) + (\alpha_1 + 2\alpha_2) = 3(\alpha_1 + \alpha_2)$ ).

On the other hand, in type  $D_n$ , the condition is still not possible when  $l = 3$ . To see this, suppose on the contrary that  $3\sigma = \gamma_1 + \gamma_2 + \gamma_3 = \gamma$ . If  $n_{\gamma_i, \beta} \leq 1$  for each  $\gamma_i$  and all  $\beta \in \Pi$ , then we are done as in type  $A_n$ . Suppose now that  $n_{\gamma_1, \beta} = 2$  for some  $\beta \in \Pi$ . If  $\eta$  is a positive root with  $n_{\eta, \beta} = 2$ , then (in the standard Bourbaki ordering)

$$\eta = \alpha_i + \cdots + \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$$

for some  $j \geq i$ . Hence, we have  $n_{\gamma_1, \alpha_{n-2}} = 2$ ,  $n_{\gamma_1, \alpha_{n-1}} = 1$  and  $n_{\gamma_1, \alpha_n} = 1$ . In order to have  $n_{\gamma, \alpha_{n-1}} = 3 = n_{\gamma, \alpha_n}$ , we must also have  $n_{\gamma_2, \alpha_{n-1}} = n_{\gamma_2, \alpha_n} = n_{\gamma_3, \alpha_{n-1}} = n_{\gamma_3, \alpha_n} = 1$ . To have  $n_{\gamma, \alpha_{n-2}}$  divisible by 3, there are two cases to consider: either  $n_{\gamma_2, \alpha_{n-2}} = 1$  and  $n_{\gamma_3, \alpha_{n-2}} = 0$  or  $n_{\gamma_2, \alpha_{n-2}} = 2 = n_{\gamma_3, \alpha_{n-2}}$ . However, there are no such roots  $\gamma_3$  satisfying  $n_{\gamma_3, \alpha_{n-2}} = 0$ ,  $n_{\gamma_3, \alpha_{n-1}} = 1$ , and  $n_{\gamma_3, \alpha_n} = 1$ . So the first case is not possible.

Suppose the latter case holds. Then for each  $i$ ,  $n_{\gamma_i, \alpha_{n-3}} \in \{1, 2\}$ . If these numbers are not all 2 or all 1, then we are done. If each  $n_{\gamma_i, \alpha_{n-3}} = 2$ , inductively computing  $n_{\gamma_i, \beta}$  for  $\beta = \alpha_m$  with  $m < n - 3$  either we are done or we come to the case that  $n_{\gamma_i, \beta} = 1$  for each  $i$ . Continuing from that case, since the  $\gamma_i$  are distinct, there is some  $\beta' = \alpha_{m'}$  with  $m' < m$  such that  $n_{\gamma_1, \beta'} = 1$ ,  $n_{\gamma_2, \beta'} \in \{0, 1\}$ , and  $n_{\gamma_3, \beta'} = 0$ . Hence,  $n_{\gamma, \beta'}$  is not divisible by 3, and we are done.

For the exceptional types, one can check “by hand”, using for example MAGMA, that the root condition  $l\sigma = \gamma_1 + \gamma_2 + \gamma_3$  cannot hold for  $l > 3$ . Hence, under our assumptions on  $l$ , we must have  $d_1(S^1(\mathbf{u}_J^*)^{[1]}) = 0$ . Since the differentials in the spectral sequence are derivations with respect to the cup product, it follows that  $d_1(S^\bullet(\mathbf{u}_J^*)^{[1]}) = 0$ .

Now we can recursively apply this argument to show that  $d_r(S^\bullet(\mathbf{u}_J^*)^{[1]}) = 0$  for  $r \geq 1$ . First observe that  $d_r(S^1(\mathbf{u}_J^*)^{[1]})$  is always a subquotient of

$$H^3(\text{gr } u_\zeta(\mathbf{u}_J), \mathbb{C}) \cong (S^1(\mathbf{u}_J^*)^{[1]} \otimes \Lambda_{\zeta, J}^1) \oplus \Lambda_{\zeta, J}^3.$$

This means that the aforementioned weight arguments can be applied. Secondly, the result after taking kernels modulo images of  $S^\bullet(\mathbf{u}_J^*)^{[1]}$  in  $E_r$  is always generated in degree one, so we can use the fact that the differentials are derivations to conclude that  $d_r(S^\bullet(\mathbf{u}_J^*)^{[1]}) = 0$ .  $\square$

We now prove that the cohomology ring contains a subalgebra isomorphic to  $S^\bullet(\mathbf{u}_J^*)^{[1]}$  by using our prior calculations in Section 4.

**Proposition 5.6.3.** *Let  $l$  satisfy Assumption 1.2.2 and  $J \subseteq \Pi$  be as in Section 4.1. There exists a subring isomorphic to  $S := S^\bullet(\mathbf{u}_J^*)^{[1]}$  contained in  $R := H^\bullet(u_\zeta(\mathbf{u}_J), \mathbb{C})$  such that  $R$  is finitely generated over  $S$ .*

PROOF. According to Proposition 5.6.2,  $d_r(S) = 0$  in the spectral sequence of Lemma 5.6.1. After taking kernels modulo images of  $S$  of the differentials  $d_r$  one obtains a quotient  $S'$  of  $S$ . Note that  $S'$  is a subring of  $R$  such that  $R$  is finitely generated over  $S'$ . Set  $M := (\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*$  as in Section 4.1. Consider the two spectral sequences:

$$E_1^{i,j} = H^{i+j}(\text{gr } u_\zeta(\mathbf{u}_J), \mathbb{C})_{(i)} \Rightarrow H^{i+j}(u_\zeta(\mathbf{u}_J), \mathbb{C}),$$

$$\tilde{E}_1^{i,j} = \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^{i+j}(\text{gr } u_\zeta(\mathbf{u}_J), \mathbb{C})_{(i)}) \Rightarrow \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^{i+j}(u_\zeta(\mathbf{u}_J), \mathbb{C})).$$

Observe that  $S = S^\bullet(\mathbf{u}_J^*)^{[1]} \subseteq \text{Hom}_{u_\zeta(\mathbf{u}_J)}(\mathbb{C}, H^\bullet(\text{gr } u_\zeta(\mathbf{u}_J), \mathbb{C}))$ . In the case when  $J \subseteq \Pi$  is as in Section 4.1, the second spectral sequence collapses and we have

$$N := \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^\bullet(\text{gr } u_\zeta(\mathbf{u}_J), \mathbb{C})) \cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, H^\bullet(u_\zeta(\mathbf{u}_J), \mathbb{C})).$$

which is a finitely generated  $S$ -module.



A description of  $N$  is given in Theorem 5.3.1, and the action of  $S$  on  $N$  is given by left multiplication on the components involving the symmetric powers of  $\mathfrak{u}_J^*$  in  $N$ . Now  $R$  is finitely generated over  $S'$  and  $S'$  is contained in  $\text{Hom}_{u_\zeta(\mathfrak{l}_J)}(\mathbb{C}, R)$  so  $S'$  will act on  $\text{Hom}_{u_\zeta(\mathfrak{l}_J)}(M, R)$ . Moreover  $N$  is a  $S'$ -summand of  $R$  because of the projectivity of  $M$ , thus  $N$  is a finitely generated  $S'$ -module. Therefore, the (Krull) dimension of  $S'$  is greater than or equal to the dimension of  $S$ . Since  $S$  is a polynomial ring and  $S'$  its quotient, we can conclude that none of the differentials in the spectral sequence could have non-zero image in  $S$ , thus  $S \cong S'$ .  $\square$

### 5.7. Proof of the main result, Theorem 1.2.3, II

We will now prove that the isomorphisms in Theorem 1.2.3 are algebra isomorphisms for those cases listed in part (b)(i). Let  $J \subseteq \Pi$  and consider the spectral sequence

$$E_2^{i,j} = R^i \text{ind}_{P_J}^G H^j(u_\zeta(\mathfrak{p}_J), \mathbb{C}) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{g}), \mathbb{C}).$$

The (vertical) edge homomorphism is a map of algebras

$$\phi : H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \rightarrow E_2^{0,\bullet},$$

where  $E_2^{0,\bullet} = \text{ind}_{P_J}^G H^\bullet(u_\zeta(\mathfrak{p}_J), \mathbb{C})$ . The map of algebras is induced by the restriction map  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \rightarrow H^\bullet(u_\zeta(\mathfrak{p}_J), \mathbb{C})$ .

Next consider the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{i,j} = H^i(u_\zeta(\mathfrak{l}_J), H^j(u_\zeta(\mathfrak{u}_J), \mathbb{C})) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{p}_J), \mathbb{C}).$$

We also have the vertical edge homomorphism  $\delta : H^\bullet(u_\zeta(\mathfrak{p}_J), \mathbb{C}) \rightarrow \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(\mathbb{C}, H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}))$ . Applying the induction functor to this map and composing with  $\phi$  yields an algebra homomorphism

$$\Psi' : H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \rightarrow \text{ind}_{P_J}^G \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(\mathbb{C}, H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

Set  $R = H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$ . From Proposition 5.6.3, there exists a subalgebra of universal cycles isomorphic to  $S^{\bullet/2}(\mathfrak{u}_J^*)^{[1]}$  in  $R$  such that  $R$  is finitely generated over the subalgebra. This subalgebra can be identified as coming from  $E_1$ -term in the spectral sequence in Lemma 5.6.1. There is an ideal  $I'$  such that  $E_1^{\bullet,\bullet}/I' \cong S^{\bullet/2}(\mathfrak{u}_J^*)^{[1]}$ . Therefore, there exists an ideal  $I$  (equivariant under  $P_J$ ) such that

$$R/I \cong S^{\bullet/2}(\mathfrak{u}_J^*)^{[1]}.$$

Consequently, there is an algebra homomorphism:

$$\Psi : H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \rightarrow \text{ind}_{P_J}^G S^{\bullet/2}(\mathfrak{u}_J^*).$$

Next we observe that our constructions are compatible with the other spectral sequences used in Sections 5.1 and 5.2. In the process of our work, we proved that there is a  $G$ -module isomorphism

$$\sigma : H^{\bullet-l(w)}(u_\zeta(\mathfrak{g}), \mathbb{C}) \rightarrow \text{ind}_{P_J}^G S^{(\bullet-l(w))/2}(\mathfrak{u}_J^*).$$

Set  $M = H^{\bullet-l(w)}(u_\zeta(\mathfrak{g}), \mathbb{C})$ . Viewing  $M$  as a module over  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  and  $\text{ind}_{P_J}^G S^{(\bullet-l(w))/2}(\mathfrak{u}_J^*)$  as a  $\text{ind}_{P_J}^G S^{\bullet/2}(\mathfrak{u}_J^*)$ -module, this isomorphism is compatible with  $\Psi$  in the sense that  $\sigma(x.y) = \Psi(x)\sigma(y)$  where  $x \in H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  and  $y \in M$ . Now suppose that  $\Psi(x) = 0$  for some  $x \in H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$ . Then  $\sigma(x.y) = 0$  for all  $y \in M$ . Set  $y = \text{id} \in H^0(u_\zeta(\mathfrak{g}), \mathbb{C})$ , so  $x = 0$  because  $\sigma$  is an isomorphism. This shows that  $\Psi$  is injective. By comparing dimensions,  $\Psi$  must be surjective, and hence an isomorphism of algebras.

## CHAPTER 6

### Finite Generation

This section provides a proof of Theorem 1.2.4 in Section 1.2. As pointed out in Chapter 1, it can also be found in [MPSW], in a more general context. Our result depends heavily on the explicit calculations of the cohomology that we achieved in Chapter 5.

#### 6.1. A finite generation result

Let  $G$  be the complex semisimple, simply connected algebraic group with root system  $\Phi$ . Maintain the notation of §2.3.

Let  $J \subseteq \Pi$ , and let  $D$  be a  $P_J$ - $S^\bullet(\mathfrak{u}_J^*)$ -module. This means that  $D$  is a rational  $P_J$ -module and a module for  $S^\bullet(\mathfrak{u}_J^*)$  such that, if  $g \in P_J$ ,  $a \in S^\bullet(\mathfrak{u}_J^*)$ , and  $x \in D$ , then  $g \cdot (ax) = (g \cdot a)(g \cdot x)$ , where we use the natural conjugation action of  $P_J$  on  $S^\bullet(\mathfrak{u}_J^*)$ .

Let  $A := \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = \mathbb{C}[G \times^{P_J} \mathfrak{u}_J]$  be the algebra of regular functions on the cotangent bundle  $G \times^{P_J} \mathfrak{u}_J$  as discussed in §3.6. Let  $\mathcal{O}$  denote the corresponding nilpotent orbit such that  $G \cdot \mathfrak{u}_J = \overline{\mathcal{O}}$  (which was denoted  $\mathcal{C}_J$  in §3.1.1). If the moment map  $\mu : G \times^{P_J} \mathfrak{u}_J \rightarrow G \cdot \mathfrak{u}_J$  is a resolution of singularities, then Lemma 3.6.1 gives that

$$\mathbb{C}[G \times^{P_J} \mathfrak{u}_J] \cong \mathbb{C}[\mathcal{O}].$$

Of course,  $\mathbb{C}[\overline{\mathcal{O}}] \subseteq \mathbb{C}[\mathcal{O}]$  and, by [Jan3, Prop. 8.3], the algebra  $\mathbb{C}[\mathcal{O}]$  is the integral closure of  $\mathbb{C}[\overline{\mathcal{O}}]$  in its field of fractions. Since  $\mathbb{C}[\overline{\mathcal{O}}] = \mathbb{C}[G \cdot \mathfrak{u}_J]$  is a finitely generated  $\mathbb{C}$ -algebra,  $\mathbb{C}[\mathcal{O}] \cong \mathbb{C}[G \times^{P_J} \mathfrak{u}_J]$  is also a finitely generated  $\mathbb{C}$ -algebra by [ZS, Chapter V, Thm. 9] (which, in fact, plays an essential role in the proof of [Jan2, Prop. 8.3]).

We now state a basic result on how finite generation is related to the induction functor which will play an important role in the discussion that follows.

**Proposition 6.1.1.** *Let  $J \subseteq \Pi$  and  $D$  be a  $P_J$ - $S^\bullet(\mathfrak{u}_J^*)$ -module which is a finitely generated  $S^\bullet(\mathfrak{u}_J^*)$ -module. If  $A := \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$  is a finitely generated  $\mathbb{C}$ -algebra, then  $R^n \text{ind}_{P_J}^G D$  is a finitely generated  $A$ -module for  $n \geq 0$ .*

**PROOF.** By assumption,  $A := \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$  is a Noetherian  $\mathbb{C}$ -algebra. Let  $G_A$  and  $(P_J)_A$  denote the group schemes obtained by extending scalars from  $\mathbb{C}$  to  $A$ . Furthermore, let  $D_A := D \otimes_{\mathbb{C}} A$  denote the corresponding  $P_J$ -module extended to a  $(P_J)_A$ -module. Since  $G/P_J$  is projective over  $\mathbb{C}$ ,  $G_A/(P_J)_A \cong (G/P_J)_A$  is projective over  $A$ . Therefore, by [Jan1, Prop. I.5.12(c)],  $R^n \text{ind}_{(P_J)_A}^{G_A} D_A$  is finitely generated over  $A$ . To complete the proof, we show that  $R^n \text{ind}_{P_J}^G D \cong R^n \text{ind}_{(P_J)_A}^{G_A} D_A$ , and we do so with the aid of sheaf cohomology.

Let  $X = G \times^{P_J} \mathfrak{u}_J$  and  $Y = G/P_J$ . Let  $Y_A := Y \times_{\mathbb{C}} \text{Spec } A$ . Because  $A = \Gamma(X, \mathcal{O}_X)$  (the space of global sections of the sheaf  $\mathcal{O}_X$ ), there is a natural morphism  $\sigma_2 : X \rightarrow \text{Spec } A \cong G \cdot \mathfrak{u}_J$ . There is also the projection morphism  $\sigma_1 : X \rightarrow Y$  which can be viewed as the cotangent bundle of  $X$ . Let  $f = \sigma_1 \times \sigma_2 : X \rightarrow Y_A$  be the pull-back morphism.

The finitely generated  $S^\bullet(\mathfrak{u}_J^*)$ -module  $D$  defines a coherent  $\mathcal{O}_X$ -module  $F = F_D$  on  $X$ . To see this, just observe that if  $V$  is an open subvariety of  $G$  of the form  $U \times P_J$  (which exists by the

Bruhat decomposition), then  $X$  contains an open affine subvariety  $V' := V \times^{P_J} \mathfrak{u}_J \cong U \times \mathfrak{u}_J$ . Then  $\Gamma(V', F) \cong \mathbb{C}[U] \otimes D$ , which is certainly a finitely generated  $\Gamma(V', \mathcal{O}_X) = \mathbb{C}[U] \otimes S^\bullet(\mathfrak{u}_J^*)$ -module.

The morphism  $f$  is an affine morphism, and it is easily verified that the direct image sheaf  $f_*F$  is a coherent  $\mathcal{O}_{Y_A}$ -module.

Thus,  $H^n(X, F) \cong H^n(Y_A, f_*F) \cong H^n(Y, \sigma_{1*}F)$  (see [Har, Ex. 4.1, p. 222]). We also use here the fact that the projection  $Y_A \rightarrow Y$  is affine. By construction,  $H^n(Y, \sigma_{1*}F) \cong R^n \operatorname{ind}_{P_J}^G D$  and  $H^n(Y_A, f_*F) \cong R^n \operatorname{ind}_{(P_J)_A}^{G_A} D_A$  (cf. [Jan1, Prop. I.5.12(a)]). Hence  $R^n \operatorname{ind}_{P_J}^G D \cong R^n \operatorname{ind}_{(P_J)_A}^{G_A} D_A$  as needed.  $\square$

### 6.2. Proof of part (a) of Theorem 1.2.4

Throughout the remainder of Section 6, we will be working under Assumption 1.2.2. Thus, Assumption 1.2.1 is in force, and, if  $\Phi$  has type  $B_n$  or  $C_n$ , then  $l > 3$ . First, we deal with the cases in which  $l \nmid n+1$  when  $\Phi$  is of type  $A_n$ ,  $l \neq 9$  when  $\Phi$  is of type  $E_6$ , and  $l \neq 7, 9$  when  $\Phi$  is of type  $E_8$ . In these cases, Theorem 1.2.3 states that the cohomology ring  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  is the coordinate algebra of the affine variety  $\mathcal{N}(\Phi_0)$ . It is therefore a finitely generated  $\mathbb{C}$ -algebra.

Next, if  $l = 7, 9$  when  $\Phi$  is of type  $E_8$ , then Theorem 1.2.3 states that

$$H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \cong \mathbb{C}[G \times^{P_J} \mathfrak{u}_J].$$

In addition, Theorem 3.6.2 says that  $\mu : G \times^{P_J} \mathfrak{u}_J \rightarrow G \cdot \mathfrak{u}_J$  is a desingularization of  $G \cdot \mathfrak{u}_J$ . The discussion above Proposition 6.1.1 in the previous subsection thus implies that  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  is a finitely generated  $\mathbb{C}$ -algebra.

In order to handle the other cases (i.e.,  $l \mid n+1$  when  $\Phi$  is of type  $A_n$ , or  $l = 9$  when  $\Phi$  is of type  $E_6$ ), we need to invoke a more general argument. Set  $A = \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$ . In each case, Theorem 1.2.3 implies that  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  has the form  $\operatorname{ind}_{P_J}^G D$ , where  $D$  is a  $P_J$ - $S^\bullet(\mathfrak{u}_J^*)$ -module which is a finitely generated  $S^\bullet(\mathfrak{u}_J^*)$ -module. Consequently, by Proposition 6.1.1,  $\operatorname{ind}_{P_J}^G D$  is a finitely generated  $A$ -module. The action of  $A$  on  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  is by way of the spectral sequence:

$$E_2^{i,j} = R^i \operatorname{ind}_{P_J}^G H^j(u_\zeta(\mathfrak{p}_J), \mathbb{C}) \Rightarrow H^{i+j}(u_\zeta(\mathfrak{g}), \mathbb{C})$$

where  $A$  is identified as a subring of universal cycles in the bottom of the filtration for the cohomology (cf. Section 5.7). Hence,  $H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  is finitely generated over  $A$ , and thus a finitely generated  $\mathbb{C}$ -algebra by the Hilbert Basis Theorem.

### 6.3. Proof of part (b) of Theorem 1.2.4

We now prove part (b) of Theorem 1.2.4. Let  $M$  be a finite dimensional  $u_\zeta(\mathfrak{g})$ -module. Without a loss of generality, we may assume that  $M$  is an irreducible  $u_\zeta(\mathfrak{g})$ -module because of the following proposition which is easily proved by using induction on the composition length of the module and the long exact sequence in cohomology.

**Proposition 6.3.1.** *Let  $R := H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$  and  $M$  be a finite dimensional  $u_\zeta(\mathfrak{g})$ -module. Suppose that  $H^\bullet(u_\zeta(\mathfrak{g}), S)$  is finitely generated over  $R$  for all irreducible  $u_\zeta(\mathfrak{g})$ -modules  $S$ . Then  $H^\bullet(u_\zeta(\mathfrak{g}), M)$  is finitely generated over  $R$ .*

Let  $S$  be an irreducible  $u_\zeta(\mathfrak{g})$ -module. Since  $S$  lifts to a  $U_\zeta(\mathfrak{g})$ -module, there exists a spectral sequence of  $R = \operatorname{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$ -modules (obtained in a manner analogous to that of Theorem 5.1.1):

$$E_2^{i,j} = R^i \operatorname{ind}_{P_J}^G H^j(u_\zeta(\mathfrak{p}_J), \operatorname{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \otimes S) \Rightarrow H^{i+j-\ell(w)}(u_\zeta(\mathfrak{g}), S).$$

Using this spectral sequence, it suffices to show that  $D := H^\bullet(u_\zeta(\mathfrak{p}_J), \text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0 \otimes S)$  is finitely generated over  $S^\bullet(\mathfrak{u}_J^*)^{[1]}$  because then  $E_2^{i,\bullet}$  is finitely generated over  $R$  for each  $i$  by Proposition 6.1.1. Moreover, this spectral sequence stops (i.e.,  $E_r = E_\infty$  for  $r$  sufficiently large) because the higher right derived functors  $R^i \text{ind}_{P_J}^G -$  vanish when  $i > \dim G/P_J$ . Thus  $E_\infty$  is finitely generated over  $R$ , and  $H^\bullet(u_\zeta(\mathfrak{g}), S)$  is finitely generated over  $R$ .

To show that  $D$  is finitely generated over  $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ , observe that one can use the LHS spectral sequence (Lemma 2.8.1) to show that

$$D \cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^*, H^\bullet(u_\zeta(\mathfrak{u}_J), S)).$$

By the same principles as used in the proposition above, it suffices to show finite generation when  $S$  is an irreducible  $u_\zeta(\mathfrak{p}_J)$ -module. Note that irreducible  $u_\zeta(\mathfrak{p}_J)$ -modules are obtained by inflating irreducible  $u_\zeta(\mathfrak{l}_J)$ -modules, so  $u_\zeta(\mathfrak{u}_J)$  acts trivially on  $S$ . Now we have

$$D \cong \text{Hom}_{u_\zeta(\mathfrak{l}_J)}((\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* \otimes S^*, H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})).$$

But,  $(\text{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{p}_J)} w \cdot 0)^* \otimes S^*$  is a projective  $u_\zeta(\mathfrak{l}_J)$ -module. Thus we need to show that  $D_P := \text{Hom}_{u_\zeta(\mathfrak{l}_J)}(P, H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}))$  is finitely generated over  $S^\bullet(\mathfrak{u}_J^*)^{[1]}$  where  $P$  is an arbitrary projective indecomposable  $u_\zeta(\mathfrak{l}_J)$ -module. Observe that  $H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \cong \bigoplus_P D_P$  where the sum is taken over all projective indecomposable  $u_\zeta(\mathfrak{l}_J)$ -modules. The claim now follows from Proposition 5.6.3 which says that  $H^\bullet(u_\zeta(\mathfrak{u}_J), \mathbb{C})$  is a finitely generated  $S^\bullet(\mathfrak{u}_J^*)^{[1]}$ -module.



## CHAPTER 7

# Comparison with Positive Characteristic

In this chapter, we consider the extent to which the methods used in computing  $H^\bullet(u_\zeta, \mathbb{C})$  can be adapted to the calculation of the cohomology algebra of the restricted enveloping algebra of a reductive algebraic group over a field of positive characteristic.

### 7.1. The setting

Let  $F$  be an algebraically closed field of positive characteristic  $p$ . In this chapter (and contrary to previous notation)  $G$  denotes a connected, simple, simply connected algebraic group defined over the prime field  $\mathbb{F}_p$ .<sup>1</sup> Fix a maximal torus  $T$ , and let  $\Phi$  be the root system of  $T$  acting on the Lie algebra  $\mathfrak{g}_F$  of  $G$ . As usual, fix a set  $\Pi$  of simple roots. The standard notational conventions for the complex Lie algebra in earlier chapters apply equally in the present case. However, the Lie algebra  $\mathfrak{g}_F$  has an extra structure provided by a “restriction map”  $x \mapsto x^{[p]}$ ; we let  $u = u(\mathfrak{g}_F)$  be the corresponding restricted enveloping algebra. Thus,  $u$  is a finite dimensional (cocommutative) Hopf algebra.

Let  $\text{Fr} : G \rightarrow G$  be the Frobenius morphism and let  $G_1$  be its scheme-theoretic kernel. It is well-known that the category of rational  $G_1$ -modules is equivalent to the category of restricted  $\mathfrak{g}_F$ -modules (i.e., to the category of  $u$ -modules). For this and historical reasons, we will state the results below in terms of the infinitesimal group scheme  $G_1$ , though the reader can replace each  $G_1$  by  $u$  if desired.

For a rational  $G$ -module  $M$ , let  $M^{(1)}$  denote the rational  $G$ -module obtained by making  $g \in G$  act on  $M$  through  $\text{Fr}(g)$ . In particular,  $G_1$  (or  $\mathfrak{g}_F$ ) acts trivially on  $M^{(1)}$ . Conversely, if  $N$  is a rational  $G$ -module on which  $G_1$  (or  $\mathfrak{g}_F$ ) acts trivially, there exists a rational  $G$ -module  $M$ , uniquely defined up to isomorphism, such that  $M^{(1)} \cong N$ . In this case, we can simply write  $M = N^{(-1)}$ .

As usual, let  $\mathcal{N} = \mathcal{N}(\mathfrak{g}_F) \subset \mathfrak{g}_F$  be the nullcone of  $\mathfrak{g}_F$ , the closed subvariety consisting of nilpotent elements (equivalently the closure of the Richardson class of regular nilpotent elements). The restricted nullcone  $\mathcal{N}_1 = \mathcal{N}_1(\mathfrak{g}_F)$  is the closed subvariety of  $\mathcal{N}$  consisting of those  $x \in \mathcal{N}$  satisfying  $x^{[p]} = 0$ . It is an irreducible variety which can be explicitly described as the closure of a specific Richardson orbit; see [CLNP] and [UGA2]. In particular,  $\mathcal{N}_1 = \mathcal{N}$  if and only if  $p \geq h$ , where  $h$  is the Coxeter number of  $G$  (i. e., the maximum of the Coxeter numbers of the various simple components of the derived subgroup  $G'$  of  $G$ ).

When  $p > h$ , it is known that  $H^\bullet(G_1, F) = H^{2\bullet}(G_1, F)$  is isomorphic as a rational  $G$ -algebra to  $F[\mathcal{N}]^{(1)}$ . This result was first proved by Friedlander-Parshall [FP2] for  $p \geq 3(h-1)$ , and then the bound was improved to  $p > h$  by Andersen and Jantzen [AJ] by different methods. Also, [AJ] provided some ad hoc calculations of  $G_1$ -cohomology in the cases when  $p \leq h$ . We will demonstrate how these calculations fit into our general framework. As noted in the introduction (with references), it has been shown that  $\text{Spec } H^{2\bullet}(G_1, F)$  is homeomorphic (as a topological space) to  $\mathcal{N}_1$ .

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<sup>1</sup>The results in this chapter can be extended to general reductive groups. We leave this issue to the interested reader.

## 7.2. Assumptions

For  $J \subset \Pi$ , we formulate two assumptions. The first assumption involves Grauert-Riemenschneider vanishing which is known to hold in the cases when  $\Phi_J = \emptyset$  or  $\Phi_J$  is of type  $A_1$  (in positive characteristic). In positive characteristic it has also been verified in a few other special cases which will be discussed in Section 7.4 below.

(A1)  $R^i \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = 0$  for  $i > 0$ .

This condition can be reformulated in the following fashion:

(A1)'  $R^i \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) \otimes \lambda = 0$  for  $i > 0$  and  $\lambda \in X(P_J)_+$ . (Here  $P_J$  is the parabolic subgroup of  $G$  with Lie algebra  $\mathfrak{p}_J$  and  $X(P_J)_+ = X(P_J) \cap X_+$ , the set of dominant weights on  $T$  which extend to define characters on  $P_J$ .)

The second assumption on  $J$  is a condition on the normality of the closure of the Richardson orbit defined by  $J$ :

(A2) The Richardson orbit closure  $G \cdot \mathfrak{u}_J$  is normal.

In the calculation of  $H^\bullet(u_\zeta, \mathbb{C})$ , assumption (A1) held generally and assumption (A2) held for the relevant subsets  $J \subset \Pi$ . However, in the positive characteristic case of this chapter, much less is known about the validity of (A1) and (A2). The situation for (A2) is as follows:

- (1) In type  $A$ , all orbits are Richardson orbits, and all orbit closures are normal (Donkin [D]).
- (2) Assume that  $J = \{\alpha\}$ ,  $\alpha \in \Pi$ , consists of a single root. Then  $G \cdot \mathfrak{u}_J$  is the closure of the so-called subregular class  $\mathcal{O}_{\text{subreg}}$  in  $\mathcal{N}$ . (It is independent of the choice of simple root  $\alpha$  [Hum1, Thm. 5.7].) Then  $G \cdot \mathfrak{u}_J = \overline{\mathcal{O}_{\text{subreg}}}$  is normal (Kumar, Lauritzen, and Thomsen [KLT]).
- (3) Generalizing (2) in some sense, assume that  $J \subseteq \Pi$  consists of pairwise orthogonal short simple roots, the corresponding Richardson orbit closure  $G \cdot \mathfrak{u}_J$  is normal (Thomsen [Th, Prop. 7]).
- (4) Finally, Christophersen [C] has recently determined the nilpotent orbits for type  $E_6$  with  $p \geq 5$  which have normal orbit closure. See below for more specific information.

## 7.3. Consequences

Using assumptions (A1) and (A2), we can determine when the cohomology algebra  $H^\bullet(G_1, F)$  identifies with the coordinate algebra  $F[\mathcal{N}_1]$  of the restricted nullcone. In the present case, the proof is simpler than that used in the calculation of  $H^\bullet(u_\zeta, \mathbb{C})$  due, in part, to the fact that the exterior algebra  $\Lambda^\bullet(\mathfrak{u}_J^*)$  has a natural structure as a  $P_J$ -module, whereas the quantized exterior algebra  $\Lambda_{\zeta, J}^\bullet$  does not (apparently) admit a natural structure as a  $U_\zeta(\mathfrak{p}_J)$ -module.

**Theorem 7.3.1.** *Let  $G$  be a connected, simple, simply connected algebraic group over an algebraically closed field  $F$  of positive characteristic  $p$  as above. Assume that  $p \geq 3$ , and that  $p$  is a very good prime for  $G$ . Let  $w \in W$  such that  $w(\Phi_0^+) = \Phi_J^+$ . Then the following statements hold.*

- (a) *If  $J \subseteq \Pi$  satisfies (A1), then*
  - (i)  $H^{2\bullet}(G_1, F)^{(-1)} \cong \text{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*)$ ;
  - (ii)  $H^{2\bullet+1}(G_1, F) = 0$ .
- (b) *If  $J \subseteq \Pi$  satisfies (A1) and (A2), then*
  - (i)  $H^{2\bullet}(G_1, F)^{(-1)} \cong F[\mathcal{N}(\Phi_0)]$ ;
  - (ii)  $H^{2\bullet+1}(G_1, F) = 0$ .

Furthermore, these identifications are isomorphisms of rational  $G$ -algebras.

PROOF. Let  $w \in W$  satisfy  $w(\Phi_0^+) = \Phi_{w \cdot 0}^+ = \Phi_J^+$ , where  $J \subseteq \Pi$ . An argument similar to that given in Chapter 5 can be applied with the functors

$$\mathrm{Hom}_{G_1}(F, \mathrm{ind}_{P_J}^G(-)) \text{ and } \mathrm{ind}_{P_J/(P_J)_1}^{G/G_1}(\mathrm{Hom}_{(P_J)_1}(F, -))$$

(from  $P_J$ -mod to  $G/G_1$ -mod) to construct a first quadrant spectral sequence

$$(7.3.1) \quad E_2^{i,j} = [R^i \mathrm{ind}_{P_J}^G(H^j((P_J)_1, \mathrm{ind}_B^{P_J} w \cdot 0)^{(-1)})]^{(1)} \Rightarrow H^{i+j-\ell(w)}(G_1, F).$$

Next, applying the Lyndon-Hochschild-Serre spectral sequence and the fact that  $\mathrm{ind}_B^{P_J} w \cdot 0$  is an injective  $(L_J)_1$ -module, we conclude that

$$(7.3.2) \quad \mathrm{Hom}_{(L_J)_1}(F, \mathrm{ind}_B^{P_J} w \cdot 0 \otimes H^j((U_J)_1, F)) = H^j((P_J)_1, \mathrm{ind}_B^{P_J} w \cdot 0).$$

In order to compute  $H^\bullet((U_J)_1, F)$ , there is a first quadrant spectral sequence of  $P_J$ -modules [FP1, Prop. 1.1] (cf. also [Jan1, I 9.20], which can be re-indexed):

$$(7.3.3) \quad E_2^{2i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes H^j(\mathfrak{u}_J, F) \Rightarrow H^{2i+j}((U_J)_1, F).$$

Here  $H^j(\mathfrak{u}_J, F)$  is the ordinary cohomology of the Lie algebra  $\mathfrak{u}_J$  or equivalently the cohomology  $H^j(\mathbb{U}(\mathfrak{u}_J), F)$  of the universal enveloping algebra of  $\mathfrak{u}_J$ . Since  $\mathrm{ind}_B^{P_J} w \cdot 0$  is an injective  $(L_J)_1$ -module, we can compose the spectral sequence in (7.3.3) with  $\mathrm{Hom}_{(L_J)_1}(F, \mathrm{ind}_B^{P_J} w \cdot 0 \otimes -)$  and use (7.3.2) to construct a spectral sequence:

$$(7.3.4) \quad E_2^{2i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{Hom}_{(L_J)_1}(F, \mathrm{ind}_B^{P_J} w \cdot 0 \otimes H^j(\mathfrak{u}_J, F)) \Rightarrow H^{2i+j}((P_J)_1, \mathrm{ind}_B^{P_J} w \cdot 0).$$

Now using the same analysis as in Chapter 4 with the Steinberg module, we can conclude that (analogous to Theorem 4.3.1)

$$\mathrm{Hom}_{(L_J)_1}(F, \mathrm{ind}_B^{P_J} w \cdot 0 \otimes H^j(\mathfrak{u}_J, F)) \cong \begin{cases} F & \text{if } j = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the spectral sequence (7.3.4) collapses to a single horizontal row and yields:

$$H^i((P_J)_1, \mathrm{ind}_B^{P_J} w \cdot 0) \cong S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}_J^*)^{(1)}.$$

By using this isomorphism and assumption (A1), the spectral sequence (7.3.1) collapses to a single horizontal row, and we obtain part (a) of the theorem. For part (b), we simply use (A2) and Theorem 3.6.2.  $\square$

Consider the case not covered in the preceding theorem which happens only when  $\Phi$  is of type  $A_n$ . This result encompasses the computation [AJ, §6.A] when  $p = n + 1$  (corresponding to  $m = 0$ ).

**Theorem 7.3.2.** *Let  $G = \mathrm{SL}_{n+1}(F)$ , where  $F$  is an algebraically closed field of positive characteristic  $p$ . Assume that  $p \geq 3$  and  $p \mid n + 1$  with  $n + 1 = p(m + 1)$ . Let  $w \in W$  be defined as in (4.8.1) by taking  $l = p$ . Then  $w(\Phi_0^+) = \Phi_J^+$ . If  $J \subseteq \Pi$  satisfies (A1)', then*

- (a)  $H^{2\bullet}(G_1, F)^{(-1)} \cong \bigoplus_{t=0}^{p-1} \mathrm{ind}_{P_J}^G S^{\frac{2\bullet - (m+1)t(p-t)}{2}}(\mathfrak{u}_J^*) \otimes \varpi_{t(m+1)};$
- (b)  $H^{2\bullet+1}(G_1, F) = 0.$

Furthermore, these are all isomorphisms of rational  $G$ -algebras.

PROOF. The proof proceeds as for Theorem 7.3.1. More precisely, the discussion through (7.3.4) holds here as well. In this case, however, we apply a result analogous to part (b) of Theorem 4.3.1



and conclude that

$$\mathrm{Hom}_{(L_J)_1}(F, \mathrm{ind}_B^P w \cdot 0 \otimes H^j(\mathfrak{u}_J, F)) \cong \begin{cases} F & \text{if } j = \ell(w) \\ p\varpi_{t(m+1)} \oplus p\varpi_{(p-t)(m+1)} & \text{if } j = \ell(w) + (m+1)t(p-t) \\ & \text{for } 1 \leq t \leq (p-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the spectral sequence (7.3.4). This consists of  $(p+1)/2$  non-zero rows in degrees  $\ell(w) + (m+1)t(p-t)$  for  $0 \leq t \leq (p-1)/2$ . Since  $t(p-t)$  is even, these non-zero rows all appear in degrees having the same parity. Since the non-zero terms appear only in even degree columns, and the differential  $d_r$  has bidegree  $(r, 1-r)$ , all differentials are zero. Hence we conclude that

$$H^i((P_J)_1, \mathrm{ind}_B^P w \cdot 0) \cong \bigoplus_{t=0}^p S^{\frac{i-\ell(w)-(m+1)t(p-t)}{2}}(\mathfrak{u}_J^*)^{(1)} \otimes p\varpi_{t(m+1)}.$$

By using this isomorphism and assumption (A1)', the spectral sequence (7.3.1) collapses, and the theorem follows.  $\square$

#### 7.4. Special cases

When  $J$  consists of a single simple root the assumptions (A1) and (A2) were verified in [Th, Prop. 7, proof of Thm. 2, Lemma 14], and (A2) has also been verified in [KLT, Thm. 6]. In this case  $\mathcal{N}(\Phi_0)$  is the closure of the subregular orbit  $\mathcal{O}_{\mathrm{subreg}}$  which occurs precisely when  $p = h-1$ . Consequently, we have the following corollary.

**Corollary 7.4.1.** *Let  $G$  be a simple algebraic group over an algebraically closed field  $F$  of positive characteristic  $p$  as above. Assume that  $p = h-1$ . Then the following statements hold.*

- (a)  $H^{2\bullet}(G_1, F)^{(-1)} \cong F[\overline{\mathcal{O}_{\mathrm{subreg}}}]$ ;
- (b)  $H^{2\bullet+1}(G_1, F) = 0$ .

*These identifications are isomorphisms of rational  $G$ -algebras.*

Using [Th], it is possible to compute the cohomology algebra  $H^\bullet(G_1, F)$  for more general examples when  $J$  consists of pairwise orthogonal simple short roots. For such  $J$ , conditions (A1) and (A2) hold. Combining this with the previous corollary gives the following.

**Corollary 7.4.2.** *Let  $G$  be a simple algebraic group over an algebraically closed field  $F$  of positive characteristic  $p$  as above. Assume the following constraints on  $p$ :*

- (i)  $p > h/2$  if  $\Phi$  is of type  $A_n$ ,  $C_n$ , or  $D_n$ ,
- (ii)  $p \geq h/2$  if  $\Phi$  is of type  $B_n$ ,
- (iii)  $p \geq 7$  if  $\Phi$  is of type  $F_4$  or  $E_6$ ,
- (iv)  $p \geq 11$  if  $\Phi$  is of type  $E_7$ ,
- (v)  $p \geq 17$  if  $\Phi$  is of type  $E_8$ .

*Then the following statements hold.*

- (a)  $H^{2\bullet}(G_1, F)^{(-1)} \cong F[\mathcal{N}(\Phi_0)]$ ;
- (b)  $H^{2\bullet+1}(G_1, F) = 0$ .

*These identifications are isomorphisms of rational  $G$ -algebras.*

Recent results of Christophersen [C, Thm. 1, Example 3.15] verify (A1) and (A2) for the group  $E_6$  when  $p \geq 5$  for those subsets  $J \subseteq \Pi$  listed in Appendix A.1, which gives the following result.

**Corollary 7.4.3.** *Let  $G$  be a simple algebraic group over an algebraically closed field  $F$  having positive characteristic  $p \geq 5$  as above. Assume that  $G$  has root system of type  $E_6$ . Then the following statements hold.*

- (a)  $H^{2\bullet}(G_1, F)^{(-1)} \cong F[\mathcal{N}(\Phi_0)];$
- (b)  $H^{2\bullet+1}(G_1, F) = 0.$

Here the subset  $\Phi_0 = \Phi_{0,p}$  is explicitly described in Appendix A.1. These identifications are isomorphisms of rational  $G$ -algebras.

With the assistance of undergraduate students at the University of Wisconsin-Stout, particularly J. Mankovecky, the original computer program written by Christophersen [C, Appendix A] for root systems of type  $E_6$  has been extended to arbitrary types; see [BMMR]. With the aid of this program and the algorithm in [C, Example 3.15], condition (A1) can be verified in Type  $E_7$  when  $p \geq 7$  for those subsets  $J \subseteq \Pi$  listed in Appendix A.1.



## CHAPTER 8

### Support Varieties over $u_\zeta$ for the Modules $\nabla_\zeta(\lambda)$ and $\Delta_\zeta(\lambda)$

In this chapter we return to the small quantum group  $u_\zeta$  in characteristic 0. We apply our results on the cohomology of the small quantum group to obtain information on the support varieties of the modules  $\nabla_\zeta(\lambda)$  and  $\Delta_\zeta(\lambda)$ . We will first compute the support varieties of the (quantum) induced modules  $\nabla_\zeta(\lambda)$ . Later, using this calculation, we will show that  $\mathcal{V}_{\mathfrak{g}}(\Delta_\zeta(\lambda)) = \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  for all  $\lambda \in X_+$ .

#### 8.1. Quantum support varieties

We will assume that  $l$  satisfies Assumption 1.2.2. In what follows, we let

$$R := H^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C})_{\text{red}} = H^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) / \text{rad}(H^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C})),$$

the quotient of the (even) cohomology algebra by its (Jacobson) radical. We have proven that  $R$  is a commutative finitely generated  $\mathbb{C}$ -algebra. Moreover, if  $M$  and  $N$  are finite dimensional  $u_\zeta(\mathfrak{g})$ -modules, then  $\text{Ext}_{u_\zeta(\mathfrak{g})}^\bullet(M, N)$  is a finitely-generated  $R$ -module. Let  $J_{M,N}$  be the annihilator of the action of  $R$  on  $\text{Ext}_{u_\zeta(\mathfrak{g})}^\bullet(M, N)$ , and set  $\mathcal{V}_{\mathfrak{g}}(M, N)$  equal to the maximum ideal spectrum of  $R/J_{M,N}$ . The variety  $\mathcal{V}_{\mathfrak{g}}(M, N)$  is called a *relative support variety* of  $M$ . This variety is a closed, conical subvariety of the variety  $\mathcal{V}_{\mathfrak{g}} := \mathcal{V}_{\mathfrak{g}}(\mathbb{C}, \mathbb{C})$ . The *support variety* of  $M$  is  $\mathcal{V}_{\mathfrak{g}}(M) := \mathcal{V}_{\mathfrak{g}}(M, M)$ . We note that if  $M$  is a  $U_\zeta(\mathfrak{g})$ -module then  $\mathcal{V}_{\mathfrak{g}}(M)$  is stable under the adjoint action of  $G$ .

In the generic case (cf. Theorem 1.2.3(b)(i)) our computation of the cohomology algebra shows that  $\mathcal{V}_{\mathfrak{g}}$  identifies with  $\mathcal{N}(\Phi_0)$  where  $\mathcal{N}(\Phi_0)$  is the subvariety of  $\mathcal{N}$  defined in (1.2.2). We will prove that this also holds in the non-generic case, but we will need to use more sophisticated techniques to verify this.

#### 8.2. Lower bounds on the dimensions of support varieties

Let  $\lambda \in X$  and let  $\Phi_\lambda = \{\alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in l\mathbb{Z}\}$ . Using the notation of Section 2.10, let  $\nabla_\zeta(\lambda) = H_\zeta^0(\lambda)$  denote the  $U_\zeta(\mathfrak{g})$ -module induced from the one dimensional  $U_\zeta(\mathfrak{b})$ -module  $\mathbb{C}_\lambda$  determined by the character  $\lambda$ . In what follows, we will consider  $\nabla_\zeta(\lambda)$  to be a  $u_\zeta(\mathfrak{g})$ -module by restriction.

The first observation to make is that one can find a lower bound on  $\dim \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  which is not dependent on whether  $l$  is good or bad.

**Proposition 8.2.1.** *Let  $\lambda \in X_+$ . Then*

$$\dim \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) \geq |\Phi| - |\Phi_\lambda|.$$

PROOF. One can use the proof given in [UGA3, §2, Corollary 2.5] by replacing  $G_1$  with  $u_\zeta(\mathfrak{g})$  (resp.  $B_1$  by  $u_\zeta(\mathfrak{b})$ ). However, we should remark that  $\mathcal{V}_{\mathfrak{b}}(\nabla_\zeta(\lambda)) \subseteq \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) \cap \mathfrak{u}$ . Since  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  is a  $G$ -variety (i.e., a union of  $G$ -orbit closures), one can use a result of Spaltenstein (cf. [Hum1, Proposition 6.7]) to conclude that  $\dim \mathcal{V}_{\mathfrak{b}}(\nabla_\zeta(\lambda)) \leq \frac{1}{2} \dim \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$ . For the quantum case, one should now replace the last line in [UGA3, Corollary 2.5] by

$$\dim \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) \geq 2 \dim \mathcal{V}_{\mathfrak{b}}(\nabla_\zeta(\lambda)) \geq 2(|\Phi^+| - |\Phi_\lambda^+|) = |\Phi| - |\Phi_\lambda|.$$

□

### 8.3. Support varieties of $\nabla_\zeta(\lambda)$ : general results

We can now present a result which allows one to compute the supports of induced/Weyl modules  $\nabla_\zeta(\lambda), \Delta_\zeta(\lambda)$  provided that it is possible to  $W$ -conjugate the stabilizer set  $\Phi_\lambda$  into a subroot system  $\Phi_J$  which is generated by a set  $J$  of simple roots. The methods used will be those provided in [NPV, Sections 5 and 6]. We outline some of the details of these arguments. For  $\lambda \in X$ , let  $Z_\zeta(\lambda) = \text{ind}_{u_\zeta(\mathfrak{b})}^{u_\zeta(\mathfrak{g})} \lambda$ . These are finite dimensional modules having dimension  $l^{|\Phi^+|}$  and are quantum analogs of the “baby Verma modules”. Moreover, for  $\lambda \in X_+$ , let  $L_\zeta(\lambda) = \text{soc}_{U_\zeta(\mathfrak{g})} \nabla_\zeta(\lambda)$  be the finite-dimensional irreducible  $U_\zeta(\mathfrak{g})$ -module of highest weight  $\lambda$ .

**Proposition 8.3.2.** *Let  $\lambda \in X_+$ ,  $w \in W$ . Then*

- (i)  $\mathcal{V}_{\mathfrak{g}}(L_\zeta(\lambda)) \subseteq G \cdot \mathcal{V}_{\mathfrak{g}}(Z_\zeta(w \cdot \lambda))$ ;
- (ii)  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) \subseteq G \cdot \mathcal{V}_{\mathfrak{g}}(Z_\zeta(w \cdot \lambda))$ .

PROOF. For (i), we apply the arguments given in [NPV, Section 5]. Using the proofs one can show that

$$\mathcal{V}_{\mathfrak{g}}(H_\zeta^{l(w)}(w \cdot \lambda)) \subseteq G \cdot \mathcal{V}_{\mathfrak{g}}(Z_\zeta(w \cdot \lambda))$$

for all  $w \in W$ . Note that these proofs make use of the relative support varieties  $\mathcal{V}_{\mathfrak{g}}(M, N)$ . With the aforementioned inclusion of support varieties, (ii) follows by using induction on the ordering of dominant weight (cf. [NPV, (5.6.1) Theorem]). □

**Proposition 8.3.3.** *Let  $\mu \in X$  such that  $\langle \mu + \rho, \alpha^\vee \rangle \in \mathbb{Z}$  for all  $\alpha \in J$ . Then*

$$\mathcal{V}_{\mathfrak{g}}(Z_\zeta(\mu)) \subseteq \mathcal{V}_{\mathfrak{u}_J}.$$

PROOF. Let  $\mathfrak{p}_J$  be the parabolic subalgebra associated to  $J$  where  $\mathfrak{p}_J \cong \mathfrak{l}_J \oplus \mathfrak{u}_J$ . Here  $\mathfrak{l}_J$  is the Levi subalgebra and  $\mathfrak{u}_J$  is the unipotent radical of  $\mathfrak{p}_J$ . Set  $Z_\zeta^J(\mu) = \text{ind}_{u_\zeta(\mathfrak{b}_J)}^{u_\zeta(\mathfrak{l}_J)} \mu$  where  $\mathfrak{b}_J$  is a Borel subalgebra of  $\mathfrak{l}_J$ . We can make  $Z_\zeta^J(\mu)$  into a  $u_\zeta(\mathfrak{p}_J)$ -module by letting  $u_\zeta(\mathfrak{u}_J)$  act trivially. Then

$$Z_\zeta(\mu) \cong \text{ind}_{u_\zeta(\mathfrak{p}_J)}^{u_\zeta(\mathfrak{g})} Z_\zeta^J(\mu).$$

By the standard Frobenius reciprocity argument (cf. [NPV, Prop. (2.3.1)]),  $\mathcal{V}_{\mathfrak{g}}(Z_\zeta(\mu)) \subseteq \mathcal{V}_{\mathfrak{p}_J}(Z_\zeta^J(\mu))$ . For any  $u_\zeta(\mathfrak{g})$ -module  $M$ , we have the following commutative diagram

$$(8.3.1) \quad \begin{array}{ccc} H^{2\bullet}(u_\zeta(\mathfrak{p}_J), \mathbb{C}) & \xrightarrow{\text{res}} & H^{2\bullet}(u_\zeta(\mathfrak{u}_J), \mathbb{C}) \\ \gamma \downarrow & & \downarrow \delta \\ \text{Ext}_{u_\zeta(\mathfrak{p}_J)}^\bullet(M, M) & \xrightarrow{\beta} & \text{Ext}_{u_\zeta(\mathfrak{u}_J)}^\bullet(M, M) \end{array}$$

Here  $\gamma = - \otimes M$ ,  $\delta = - \otimes M$ , and the bottom horizontal restriction map is labeled  $\beta$ . Now set  $M := Z_\zeta^J(\mu)$ . The action of  $u_\zeta(\mathfrak{u}_J)$  on  $Z_\zeta^J(\mu)$  is trivial, thus  $\text{Ext}_{u_\zeta(\mathfrak{u}_J)}^\bullet(M, M) \cong \text{Ext}_{u_\zeta(\mathfrak{u}_J)}^\bullet(\mathbb{C}, \mathbb{C}) \otimes M^* \otimes M$ . This shows that  $\delta$  is an injection.

Under the hypothesis of the proposition (i.e.,  $\langle \mu + \rho, \alpha^\vee \rangle \in \mathbb{Z}$  for all  $\alpha \in J$ ), we can conclude that  $Z_\zeta^J(\mu)$  is projective as a  $u_\zeta(\mathfrak{l}_J)$ -module. By applying the Lyndon-Hochschild-Serre spectral sequence for  $u_\zeta(\mathfrak{u}_J)$  normal in  $u_\zeta(\mathfrak{p}_J)$ , we have

$$\text{Ext}_{u_\zeta(\mathfrak{p}_J)}^\bullet(M, M) \cong \text{Ext}_{u_\zeta(\mathfrak{u}_J)}^\bullet(M, M)^{u_\zeta(\mathfrak{l}_J)}.$$

This also shows that  $\beta$  is injective. Therefore,  $\ker(\text{res}) = \ker(\gamma)$ . By definition  $\mathcal{V}_{\mathfrak{p}_J}(Z_\zeta^J(\mu))$  is the variety associated to the ideal  $\ker(\gamma)$ , and the image of the map  $\mathcal{V}_{\mathfrak{u}_J} \rightarrow \mathcal{V}_{\mathfrak{p}_J}$  is given by the ideal  $\ker(\text{res})$ . It follows that  $\mathcal{V}_{\mathfrak{p}_J}(Z_\zeta^J(\mu)) = \mathcal{V}_{\mathfrak{u}_J}$ . □

The following two results (generalizing [NPV, (6.2.1) Thm.]) provide a description of the support varieties of the modules  $\nabla_\zeta(\lambda)$  for  $\lambda \in X_+$  in terms of closures of Richardson orbits. This is Theorem 1.2.5. Observe that there are restrictions on  $l$  even in the case when  $l > h$ .

**Theorem 8.3.4.** *Let  $\lambda \in X_+$ . Suppose there exists  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_J$ , then*

$$\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = G \cdot \mathfrak{u}_J.$$

PROOF. Let  $w \in W$  such that  $\Phi_{w \cdot \lambda} = w(\Phi_\lambda) = \Phi_J$ , and  $\mathfrak{u}_J \subseteq \mathcal{N}(\Phi_0)$ . Moreover,

$$H^{2\bullet}(u_\zeta(\mathfrak{u}_J), \mathbb{C})_{\text{red}} \cong S^\bullet(\mathfrak{u}_J^*)^{[1]}$$

by using the proofs in Section 5.6 and 5.7. Therefore,  $\mathcal{V}_{\mathfrak{u}_J} \cong \mathfrak{u}_J$ .

According to Proposition 8.3.2(ii) and Proposition 8.3.3:

$$(8.3.2) \quad \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) \subseteq G \cdot \mathcal{V}_{\mathfrak{g}}(Z_\zeta(w \cdot \lambda)) \subseteq G \cdot \mathfrak{u}_J.$$

Now  $G \cdot \mathfrak{u}_J$  is an irreducible variety of dimension equal to  $|\Phi| - |\Phi_J| = |\Phi| - |\Phi_\lambda|$ . The statement of the theorem now follows by Proposition 8.2.1.  $\square$

**Corollary 8.3.5.** *Let  $\mathfrak{g}$  be a complex simple Lie algebra and  $l$  be an odd positive integer which is good for  $\Phi$ . If  $\lambda \in X_+$ , then there exists  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_J$  for some  $J \subseteq \Pi$ , and  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = G \cdot \mathfrak{u}_J$ .*

PROOF. This follows immediately from Lemma 3.1.1 and Theorem 8.3.4.  $\square$

#### 8.4. Support varieties of $\Delta_\zeta(\lambda)$ when $l$ is good

We will now show that  $\mathcal{V}_{\mathfrak{g}}(\Delta_\zeta(\lambda)) = \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  for all  $\lambda \in X_+$ . Recall that  $L_\zeta(\lambda)$  is the irreducible module which appears as the socle of  $\nabla_\zeta(\lambda)$ .

**Theorem 8.4.6.** *Let  $\mathfrak{g}$  be a complex simple Lie algebra and  $l$  be an odd positive integer which is good for  $\Phi$ . Let  $\lambda \in X_+$ , and choose  $J \subseteq \Pi$  such that  $w(\Phi_\lambda) = \Phi_J$  for some  $w \in W$ . Then*

- (a)  $\mathcal{V}_{\mathfrak{g}}(L_\zeta(\lambda)) \subseteq G \cdot \mathfrak{u}_J$ ;
- (b)  $\mathcal{V}_{\mathfrak{g}}(\Delta_\zeta(\lambda)) = G \cdot \mathfrak{u}_J$ .

PROOF. We can prove part (a) by using induction on the ordering of weights. If  $\mu$  is linked under the dot action of the affine Weyl group to  $\lambda$  and is minimal among all dominant weights  $\leq \lambda$ , then  $L_\zeta(\mu) = \nabla_\zeta(\mu)$ . Moreover,  $\Phi_\mu$  is  $W$ -conjugate to  $\Phi_\lambda$ . According to Corollary 8.3.5 and the fact that  $L_\zeta(\mu) = \nabla_\zeta(\mu)$  we have

$$\mathcal{V}_{\mathfrak{g}}(L_\zeta(\mu)) = \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\mu)) \subseteq G \cdot \mathfrak{u}_J.$$

Now assume that for all  $\mu$  linked to  $\lambda$  with  $\mu$  dominant and  $\mu < \lambda$  we have  $\mathcal{V}_{\mathfrak{g}}(L_\zeta(\mu)) \subseteq G \cdot \mathfrak{u}_J$ . There exists a short exact sequence:

$$0 \rightarrow L_\zeta(\lambda) \rightarrow \nabla_\zeta(\lambda) \rightarrow N \rightarrow 0$$

where  $N$  has composition factors  $< \lambda$  and linked to  $\lambda$ . Therefore,  $\mathcal{V}_{\mathfrak{g}}(N) \subseteq G \cdot \mathfrak{u}_J$  (by the induction hypothesis). If  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is a short exact sequence of finite-dimensional  $u_\zeta(\mathfrak{g})$ -modules, it is an elementary fact that  $\mathcal{V}_{\mathfrak{g}}(M_{\sigma(1)}) \subseteq \mathcal{V}_{\mathfrak{g}}(M_{\sigma(2)}) \cup \mathcal{V}_{\mathfrak{g}}(M_{\sigma(3)})$  for any  $\sigma \in \mathfrak{S}_3$  [PW, Lemma 5.2]. Therefore,

$$\mathcal{V}_{\mathfrak{g}}(L_\zeta(\lambda)) \subseteq \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) \cup \mathcal{V}_{\mathfrak{g}}(N) \subseteq G \cdot \mathfrak{u}_J \cup G \cdot \mathfrak{u}_J \subseteq G \cdot \mathfrak{u}_J.$$

For part (b) first observe that all composition factors of  $\Delta_\zeta(\lambda)$  have highest weights which are strongly linked to  $\lambda$  under the affine Weyl group. By part (a) these composition factors have their support varieties contained in  $G \cdot \mathfrak{u}_J$ , thus  $\mathcal{V}_{\mathfrak{g}}(\Delta_\zeta(\lambda)) \subseteq G \cdot \mathfrak{u}_J$ . Since the characters of  $\nabla_\zeta(\lambda)$  and

$\Delta_\zeta(\lambda)$  are equal, it follows that their graded dimensions are equal and one can apply the argument in Proposition 8.2.1 to show that

$$\dim \mathcal{V}_{\mathfrak{g}}(\Delta_\zeta(\lambda)) \geq |\Phi| - |\Phi_\lambda|.$$

Now by the same reasoning provided in Theorem 8.3.4, we have  $\mathcal{V}_{\mathfrak{g}}(\Delta_\zeta(\lambda)) = G \cdot u_J$ .

□

### 8.5. A question of naturality of support varieties

One fact that has not been established in the quantum setting is the “naturality” of support varieties. In particular for  $M$  a  $u_\zeta(\mathfrak{g})$ -module we would like to have the following statement:

$$(8.5.3) \quad \mathcal{V}_{\mathfrak{b}}(M) = \mathcal{V}_{\mathfrak{g}}(M) \cap \mathcal{V}_{\mathfrak{b}}.$$

In fact, one inclusion is true:

$$(8.5.4) \quad \mathcal{V}_{\mathfrak{b}}(M) \subseteq \mathcal{V}_{\mathfrak{g}}(M) \cap \mathcal{V}_{\mathfrak{b}}.$$

The equality of support varieties (8.5.3) has been established in the restricted Lie algebra setting because of the description of support varieties via rank varieties. We anticipate that this will eventually be established for quantum groups.

Throughout the remainder of the paper, we will make the following assumption on naturality of support varieties for  $\mathcal{V}_{\mathfrak{g}}(\nabla(\lambda))$ .

**Assumption 8.5.7.** *Given  $\lambda \in X_+$ ,  $\mathcal{V}_{\mathfrak{b}}(\nabla(\lambda)) = \mathcal{V}_{\mathfrak{g}}(\nabla(\lambda)) \cap \mathcal{V}_{\mathfrak{b}}$ .*

Clearly, the validity of (8.5.3) implies the validity of Assumption 8.5.7. One key result which is needed for analyzing the bad  $l$  situation is a quantum analogue of an important result of Jantzen [Jan4, Prop. (2.4)]; see also [NPV, Cor. (4.5.1)]. In its statement, given  $J \subseteq \Pi$ , we set  $x_J := \sum_{\alpha \in J} x_\alpha$ , where  $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ . We can take  $x_\emptyset = 0$ , by definition.

**Proposition 8.5.8.** *Assume that Assumption 8.5.7 is valid. Let  $J \subseteq \Pi$  and  $\lambda \in X_+$ . If  $w(\Phi_\lambda) \cap \Phi_J \neq \emptyset$  for all  $w \in W$ , then  $x_J \notin \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$ .*

In order to prove this proposition, one can apply the machinery set up in [NPV, Section 4] involving relative support varieties and “baby” Verma module. However, in the proof of [NPV, Cor. (4.5.1)], the statement of Assumption 8.5.7 is a key ingredient in the proof.

### 8.6. The Constrictor Method I

In order to compute support varieties  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  when  $l$  is a bad and there does not exist  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_J$  for some  $J \subseteq \Pi$ , we need to utilize the “constrictor method” as presented in [UGA3] to obtain upper bounds on the support varieties. Let  $\mathcal{O}$  be an orbit in  $\mathcal{N}(\Phi_0)$ . The *constrictors* of  $\mathcal{O}$  are the orbits contained in  $\mathcal{N}(\Phi_0) - \overline{\mathcal{O}}$  which are minimal with respect to the closure ordering of orbits in  $\mathcal{N}$ . The following theorem holds in the quantum case as long as Assumption 8.5.7 is valid.

**Theorem 8.6.9.** *Let  $\lambda \in X_+$ . Assume that Assumption 8.5.7 holds. Let  $\mathcal{O}$  be an orbit in  $\mathcal{N}(\Phi_0)$  and  $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s\}$  be the set of constrictors of  $\mathcal{O}$ . Assume that the following conditions are satisfied:*

- (i)  $|\Phi| - |\Phi_\lambda| \geq \dim \mathcal{O}$ ;
- (ii) for  $i = 1, 2, \dots, s$ ,  $\mathcal{O}_i = G \cdot x_{J_i}$  for some  $J_i \subseteq \Pi$ ;
- (iii) for  $i = 1, 2, \dots, s$ ,  $w(\Phi_\lambda) \cap \Phi_{J_i} \neq \emptyset$  for all  $w \in W$ .

*Then  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}}$ .*

### 8.7. The Constrictor Method II

In our computations, a more powerful method is necessary to force the containment of  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda))$  inside the closure of an orbit. In the Lie algebra case it was enough to use Theorem 8.6.9 because the constrictors had orbit representatives given by sums of simple root vectors. As we will see later this is not true in the quantum case.

**Proposition 8.7.10.** *Assume that  $\mathcal{O}$  is a  $G$ -orbit in  $\mathcal{N}$ . Let  $\lambda \in X$  and assume that*

- (i) *there exists  $w \in W$  with  $w(\Phi_\lambda) = \Phi_J$  with  $J \subseteq \Pi$ ,*
- (ii)  *$|\Phi_\lambda| > \text{codim}_{\mathcal{N}} \mathcal{O}$ .*

*Then  $\mathcal{O} \cap \mathcal{V}_{\mathfrak{g}}(Z_\zeta(\lambda)) = \emptyset$ .*

PROOF. Suppose that  $\mathcal{O} \cap \mathcal{V}_{\mathfrak{g}}(Z_\zeta(\lambda)) \neq \emptyset$ . Now

$$G \cdot \mathcal{V}_{\mathfrak{g}}(Z_\zeta(\lambda)) \subseteq \bigcup \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(w' \cdot \lambda))$$

where the union is taken over  $w' \in W_l$  with  $w' \cdot \lambda \in X_+$  (cf. [NPV, Thm. (4.6.1)]). Therefore,  $\mathcal{O} \subseteq \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(y \cdot \lambda))$  for some  $y \in W_l$  because  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(y \cdot \lambda))$  is  $G$ -stable.

By assumption there exists  $w \in W$  with  $w(\Phi_\lambda) = \Phi_J$  with  $J \subseteq \Pi$ . Moreover,

$$\mathcal{V}_{\mathfrak{g}}(Z_\zeta(w \cdot \lambda)) \subseteq \mathfrak{u}_J.$$

Applying the methods in [NPV, Thm. (5.6.1)], we have

$$\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(y \cdot \lambda)) \subseteq G \cdot \mathcal{V}_{\mathfrak{g}}(Z_\zeta(w \cdot \lambda)) \subseteq G \cdot \mathfrak{u}_J.$$

Therefore,

$$\text{codim}_{\mathcal{N}} G \cdot \mathfrak{u}_J \leq \text{codim}_{\mathcal{N}} \mathcal{O} < |\Phi_\lambda|.$$

On the other hand,

$$\text{codim}_{\mathcal{N}} G \cdot \mathfrak{u}_J = |\Phi| - 2 \dim \mathfrak{u}_J = |\Phi_J| = |\Phi_\lambda|.$$

□

We can now prove a generalization of Theorem 8.6.9.

**Theorem 8.7.11.** *Let  $\mathcal{O}$  be an orbit in  $\mathcal{N}(\Phi_0)$  with  $\mathcal{O} = G \cdot \mathcal{O}_I$  where  $\mathcal{O}_I$  is an  $L_I$ -orbit for some  $I \subset \Pi$ . Let  $\lambda \in X_+$  and assume that Assumption 8.5.7 holds. Suppose that*

- (i) *there exists  $y \in W$  with  $y(\Phi_\lambda) \cap \Phi_I = \Phi_J$  with  $J \subseteq I$ ,*
- (ii)  *$|w(\Phi_\lambda) \cap \Phi_I| > \text{codim}_{\mathcal{N}(l_I)} \mathcal{O}_I$  for all  $w \in W$ .*

*Then  $\mathcal{O} \cap \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = \emptyset$ .*

PROOF. Suppose that  $|w(\Phi_\lambda) \cap \Phi_I| > \text{codim}_{\mathcal{N}(l_I)} \mathcal{O}_I$  for all  $w \in W$ . It follows from Proposition 8.7.10 that  $\mathcal{O}_I \cap \mathcal{V}_{l_I}(Z_\zeta^I(w \cdot \lambda)) = \emptyset$  for all  $w \in W$ . Therefore,  $\mathcal{O}_I \cap \mathcal{V}_{l_I}(\oplus_{w \in W} Z_\zeta^I(w \cdot \lambda)) = \emptyset$ , and  $\mathcal{O}_I \cap \mathcal{V}_{\mathfrak{g}}(\oplus_{w \in W} Z_\zeta(w \cdot \lambda)) = \emptyset$  (cf. [NPV, Prop. (4.2.1)]). Consequently,  $\mathcal{O} \cap \mathcal{V}_{\mathfrak{g}}(\oplus_{w \in W} Z_\zeta(w \cdot \lambda)) = \emptyset$ , and by Proposition 8.3.2  $\mathcal{O} \cap \mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = \emptyset$ . □

### 8.8. Support varieties of $\nabla_\zeta(\lambda)$ when $l$ is bad

Assume for the remainder of the paper that Assumption 8.5.7 holds for all  $\lambda \in X_+$ . Under the assumptions stated at the beginning of the paper, the remaining cases that we are forced to consider are the following (arranged in the order of difficulty):

- $E_6$  when  $3 \mid l$ ;
- $F_4$  when  $3 \mid l$ ;
- $E_7$  when  $3 \mid l$ ;



- $E_8$  when  $3 \mid l$ ;
- $E_8$  when  $5 \mid l$ .

Let  $\Phi_\lambda^\vee = \{\alpha^\vee : \alpha \in \Phi_\lambda\}$ . Then  $\Phi_\lambda^\vee$  is a subroot system (cf. [Ka, Defn. 12.1]) in the dual root system  $\Phi^\vee$ . If  $\Gamma$  is a subroot system of  $\Phi^\vee$  then let  $W(\Gamma)$  be the subgroup of the Weyl group (for  $\Phi^\vee$ ) generated by the reflections in  $\Gamma$ .

Our goal is to classify all  $\Phi_\lambda^\vee$  (or equivalently  $\Phi_\lambda$ ) such that  $\Phi_\lambda$  is not  $W$  conjugate to  $\Phi_J$  for any  $J \subseteq \Pi$ . In this process we will use the Borel-de Siebenthal Theorem [Ka, Thm. 12.1]:

**Theorem 8.8.12.** *Let  $\Phi$  be an irreducible root system,  $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  be a set of simple roots and  $\alpha_h = \sum_{i=1}^n h_i \alpha_i$  be the highest root of  $\Phi$ . Then the maximal closed subroot systems of  $\Phi$  (up to  $W$  conjugation) are those with the fundamental systems:*

- (i)  $\Pi - I$  where  $I$  consists of a subset of simple roots  $\alpha_i$  where  $h_i = 1$ ,
- (ii)  $\{-\alpha_h, \alpha_1, \alpha_2, \dots, \alpha_n\} - I$  where  $I$  consists of a subset of simple roots  $\alpha_i$  where  $h_i = p$  where  $p$  is a prime.

**Remark 8.8.13.** In what follows, for a given  $\lambda$ , if there does exist  $w \in W$  and  $J \subseteq \Pi$  such that  $w(\Phi_\lambda) = \Phi_J$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda))$  is determined by Theorem 8.3.4.

### 8.9. $E_6$ when $3 \mid l$

We will first proceed to classify all  $\Phi_\lambda$  (or equivalently  $\Phi_\lambda^\vee$ ) such that  $\Phi_\lambda$  is not of the form  $w(\Phi_J)$  for any  $w \in W$  and  $J \subseteq \Delta$ . One can apply the Borel-de Siebenthal Theorem which describes maximal subroot systems via the extended Dynkin diagrams. For  $E_6$  there are three possibilities:  $\Phi_\lambda^\vee$  is a subroot system of either  $D_5$ ,  $A_1 \times A_5$  or  $A_2 \times A_2 \times A_2$ .

One can immediately rule out  $D_5$  because this is a subroot system of the ordinary Dynkin diagram. In the other cases, where  $\Gamma = A_1 \times A_5$  or  $A_2 \times A_2 \times A_2$  we have  $\mathbb{Z}\Gamma/\mathbb{Z}\Phi_\lambda^\vee$  has torsion prime to  $l$  so there exists  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_I$  where  $I \subseteq \Pi(\Gamma)$ . Here  $\Pi(\Gamma)$  is a “set of simple roots” for the root system  $\Gamma$  given by the Borel-de Siebenthal Theorem.

We can analyze both cases when  $\Gamma = A_1 \times A_5$  or  $A_2 \times A_2 \times A_2$ . First note that if the “additional simple root from the extended diagram” (cf. [Ka, p. 137]) is not in  $\Phi_I$  then  $\Phi_I \subseteq \Phi$  and  $\Phi_\lambda$  is  $W$ -conjugate to  $\Phi_I$ . Therefore, we can assume that  $I$  contains the additional simple root from the extended Dynkin diagram. We can conclude that the possibilities of non-conjugate  $\Phi_\lambda$  to  $\Phi_J$  where  $J \subset \Pi$  reduces to the cases when  $\Phi_\lambda$  is possibly:

- (i)  $A_5 \times A_1$ ,
- (ii)  $A_3 \times A_1 \times A_1$ ,
- (iii)  $A_1 \times A_1 \times A_1 \times A_1$ ,
- (iv)  $A_2 \times A_2 \times A_2$ .

Further reductions can be made by arguing in the following way. For example, if  $\Phi_\lambda$  is of type  $A_5 \times A_1$  then  $\langle \lambda + \rho, \alpha_0^\vee \rangle \in 3\mathbb{Z}$ . Since  $\langle \lambda + \rho, \alpha_j^\vee \rangle \in 3\mathbb{Z}$  for  $j \neq 2$  this forces  $\langle \lambda + \rho, 2\alpha_2^\vee \rangle \in 3\mathbb{Z}$ . Therefore,  $\langle \lambda + \rho, \alpha_2^\vee \rangle \in 3\mathbb{Z}$  which is a contradiction.

If  $\Phi_\lambda$  is of type  $A_3 \times A_1 \times A_1$  then  $\langle \lambda + \rho, 2\alpha_2^\vee + 2\alpha_5^\vee \rangle \in 3\mathbb{Z}$ , thus  $\langle \lambda + \rho, \alpha_2^\vee + \alpha_5^\vee \rangle \in 3\mathbb{Z}$ . Furthermore,  $\langle \lambda + \rho, \alpha_4^\vee \rangle \in 3\mathbb{Z}$ , so  $\langle \lambda + \rho, \alpha_2^\vee + \alpha_4^\vee + \alpha_5^\vee \rangle \in 3\mathbb{Z}$  which is a contradiction. One can argue in a similar way to rule out  $\Phi_\lambda$  having type  $A_1 \times A_1 \times A_1 \times A_1$ .

This reduces us to the case when  $\Phi_\lambda$  is of type  $A_2 \times A_2 \times A_2$  which can be realized by the weight  $(l-1)(1, 1, 1, q-1, 1, 1)$  where  $l = 3q$ . Moreover, we can use the calculations and Constrictor Method I as outlined in [UGA3, 4.2] to deduce the following result.

**Theorem 8.9.14.** *Let  $\Phi$  be of type  $E_6$  and let  $3 \mid l$ . If  $\lambda \in X_+$  and  $\Phi_\lambda$  is not  $W$  conjugate to  $\Phi_J$  for any  $J \subseteq \Pi$ , then*

- (a)  $\Phi_\lambda$  is of type  $A_2 \times A_2 \times A_2$ ;
- (b)  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(A_2 \times A_1)}$ .

### 8.10. $F_4$ when $3 \mid l$

We first apply the Borel-de Siebenthal Theorem which describes maximal subroot systems containing  $\Phi_\lambda^\vee$  via the extended Dynkin diagrams. For  $F_4$  there are three possibilities:  $A_1 \times C_3$ ,  $B_4$  and  $A_2 \times A_2$ . We have to consider all three cases since the “additional simple root” in the extended Dynkin diagram is involved. In these cases,  $\mathbb{Z}\Gamma/\mathbb{Z}\Phi_\lambda^\vee$  has torsion prime to  $l$  so there exists  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_I$  where  $I \subseteq \Pi(\Gamma)$ .

We now analyze each of these cases and we only need to consider when the “additional simple root from the extended diagram” is in  $\Phi_I$ . We can conclude that the possibilities of non-conjugate  $\Phi_\lambda$  to  $\Phi_J$  where  $J \subset \Pi$  reduces to the cases when  $\Phi_\lambda$  is possibly:

- (i)  $C_3 \times A_1$ ,
- (ii)  $C_2 \times A_1$ ,
- (iii)  $A_1 \times A_1 \times A_1$ ,
- (iv)  $A_1 \times A_1$ ,
- (v)  $B_4$ ,
- (vi)  $A_3$ ,
- (vii)  $A_2 \times A_2$ .

Further reductions can be made by arguing in the following way. For example, if  $\Phi_\lambda$  is of type  $C_3 \times A_1$  then  $\langle \lambda + \rho, \alpha_h^\vee \rangle \in 3\mathbb{Z}$ . Since  $\langle \lambda + \rho, \alpha_j^\vee \rangle \in 3\mathbb{Z}$  for  $j = 2, 3, 4$  this forces  $\langle \lambda + \rho, 2\alpha_1^\vee \rangle \in 3\mathbb{Z}$ . Therefore,  $\langle \lambda + \rho, \alpha_1^\vee \rangle \in 3\mathbb{Z}$  which is a contradiction. One can argue in a similar way to rule out  $\Phi_\lambda$  in cases (ii)-(vi).

This reduces us to the case when  $\Phi_\lambda$  is of type  $A_2 \times A_2$ . One can invoke Constrictor Method I as outlined in [UGA3, 4.2] to deduce the following result.

**Theorem 8.10.15.** *Let  $\Phi$  be of type  $F_4$  and let  $3 \mid l$ . If  $\lambda \in X_+$  and  $\Phi_\lambda$  is not  $W$  conjugate to  $\Phi_J$  for any  $J \subseteq \Pi$ , then*

- (a)  $\Phi_\lambda$  is of type  $A_2 \times A_2$ ;
- (b)  $\mathcal{V}_{\mathfrak{g}}(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(A_1 \times A_2)}$ .

### 8.11. $E_7$ when $3 \mid l$

The Borel-de Siebenthal Theorem can be used to describes maximal subroot systems of  $E_7$  containing  $\Phi_\lambda^\vee$  via the extended Dynkin diagrams. There are four possibilities:  $E_6$ ,  $A_1 \times D_6$ ,  $A_7$ , and  $A_2 \times A_5$ . The  $E_6$ -case reduces us to looking at a subroot system of type  $A_2 \times A_2 \times A_2$ . For the other three cases the “additional simple root” in the extended Dynkin diagram is involved. In the these cases,  $\mathbb{Z}\Gamma/\mathbb{Z}\Phi_\lambda^\vee$  has torsion prime to  $l$  so there exists  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_I$  where  $I \subseteq \Pi(\Gamma)$ .

We now analyze each of these cases and we only need to consider when the “additional simple root from the extended diagram” is in  $\Phi_I$ . We can conclude that the possibilities of non-conjugate  $\Phi_\lambda$  to  $\Phi_J$  where  $J \subset \Pi$  reduces to the cases when  $\Phi_\lambda$  is possibly:

- (i)  $D_6 \times A_1$ ,
- (ii)  $A_1 \times A_1 \times A_3 \times A_1$ ,
- (iii)  $A_1 \times A_1 \times A_1 \times A_1$ ,
- (iv)  $D_4 \times A_1 \times A_1$ ,
- (v)  $A_7$ ,
- (vi)  $A_6$ ,
- (vii)  $A_3 \times A_3$ ,

- (viii)  $A_2 \times A_5$ ,
- (ix)  $A_2 \times A_2 \times A_2$ .

The same type of arguments given in the  $E_6$  and  $F_4$  cases reduces us to looking at  $\Phi_\lambda$  of type  $A_2 \times A_5$  and  $A_2 \times A_2 \times A_2$ . We need to now invoke the Constrictor Method II to deduce the following result.

**Theorem 8.11.16.** *Let  $\Phi$  be of type  $E_7$  and let  $3 \mid l$ . If  $\lambda \in X_+$  and  $\Phi_\lambda$  is not  $W$  conjugate to  $\Phi_J$  for any  $J \subseteq \Pi$ , then either  $\Phi_\lambda$  is of type  $A_2 \times A_5$  or  $A_2 \times A_2 \times A_2$ .*

- (a) *If  $\Phi_\lambda$  is of type  $A_2 \times A_5$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(2A_2 + A_1)}$ .*
- (b) *If  $\Phi_\lambda$  is of type  $A_2 \times A_2 \times A_2$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(A_5 + A_1)}$ .*

PROOF. For part (a), we use Constrictor Method I (with constrictor  $\mathcal{O}(A_3)$ ), and show that  $|w(\Phi_\lambda) \cap \Phi_J| > 0$  for all  $w \in W$  and  $\Phi_J \cong A_3$ . This was accomplished by using MAGMA [BC, BCP]. In order to verify part (b), we use Constrictor Method I with the orbit  $\mathcal{O}(A'_5)$  and Constrictor Method II with the orbit  $\mathcal{O}(D_5(a_1) + A_1)$ . In the latter case we verified that  $|w(\Phi_\lambda) \cap \Phi_J| > 2$  for all  $w \in W$  and  $\Phi_J \cong D_5 \times A_1$ .  $\square$

### 8.12. $E_8$ when $3 \mid l$ , $5 \mid l$

First observe that we need to handle both cases simultaneously because there are cases when both  $3 \mid l$  and  $5 \mid l$ . One can apply the Borel-de Siebenthal Theorem and for  $E_8$  there are five possibilities for maximal subroot systems:  $D_8$ ,  $A_1 \times E_7$ ,  $A_8$ ,  $A_2 \times E_6$ , and  $A_4 \times A_4$ . In all five cases, the “additional simple root” in the extended Dynkin diagram is involved. Furthermore, in the case  $D_8$ ,  $A_8$ ,  $A_4 \times A_4$ ,  $\mathbb{Z}\Gamma/\mathbb{Z}\Phi_\lambda^\vee$  has torsion prime to  $l$  so there exists  $w \in W$  such that  $w(\Phi_\lambda) = \Phi_I$  where  $I \subseteq \Pi(\Gamma)$ . For the other cases,  $A_1 \times E_7$ ,  $A_2 \times E_6$ , we rely on our prior analysis for the  $E_6$  and  $E_7$  cases.

Once again we analyze each of these cases and only need to consider when the “additional simple root from the extended diagram” is in  $\Phi_I$ . The possibilities of non-conjugate  $\Phi_\lambda$  to  $\Phi_J$  where  $J \subset \Pi$  reduces to the cases:

- (i)  $D_8$ ,
- (ii)  $D_6 \times A_1$ ,
- (iii)  $D_4 \times A_3$ ,
- (iv)  $D_4 \times A_1 \times A_1$ ,
- (v)  $A_1 \times A_1 \times A_5$ ,
- (vi)  $A_1 \times A_1 \times A_1 \times A_3$ ,
- (vii)  $A_1 \times A_1 \times A_1 \times A_1 \times A_1$ ,
- (viii)  $A_1 \times E_7$ ,
- (ix)  $A_1 \times A_2 \times A_5$ ,
- (x)  $A_1 \times A_2 \times A_2 \times A_2$ ,
- (xi)  $A_8$ ,
- (xii)  $A_5 \times A_2$ ,
- (xiii)  $A_2 \times A_2 \times A_2$ ,
- (xiv)  $A_2 \times E_6$ ,
- (xv)  $A_2 \times A_2 \times A_2 \times A_2$ ,
- (xvi)  $A_4 \times A_4$ .

We can use the prior arguments involving “finding additional roots” in  $\Phi_\lambda$  to rule out cases (i)–(ix). This reduces us to the case when  $\Phi_\lambda$  is of type (x)–(xvi). The following theorems summarize our findings.

**Theorem 8.12.17 (A).** *Let  $\Phi$  be of type  $E_8$  and let  $3 \mid l$  or  $5 \mid l$ . If  $\lambda \in X_+$  and  $\Phi_\lambda$  is not  $W$  conjugate to  $\Phi_J$  for any  $J \subseteq \Pi$ , then  $\Phi_\lambda$  is of type*

- (i)  $A_1 \times A_2 \times A_2 \times A_2$ ,
- (ii)  $A_8$ ,
- (iii)  $A_5 \times A_2$ ,
- (iv)  $A_2 \times A_2 \times A_2$ ,
- (v)  $A_2 \times E_6$ ,
- (vi)  $A_2 \times A_2 \times A_2 \times A_2$ ,
- (vii)  $A_4 \times A_4$ .

**Theorem 8.12.18 (B).** *Let  $\Phi$  be of type  $E_8$  and let  $3 \mid l$  or  $5 \mid l$ . Suppose that  $\lambda \in X_+$  and  $\Phi_\lambda$  is not  $W$  conjugate to  $\Phi_J$  for any  $J \subseteq \Pi$ .*

- (i) *If  $\Phi_\lambda$  is of type  $A_1 \times A_2 \times A_2 \times A_2$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(E_8(b_6))}$ .*
- (ii) *If  $\Phi_\lambda$  is of type  $A_8$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(2A_2 + 2A_1)}$ .*
- (iii) *If  $\Phi_\lambda$  is of type  $A_5 \times A_2$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(E_6(a_3) + A_1)}$ .*
- (iv) *If  $\Phi_\lambda$  is of type  $A_2 \times A_2 \times A_2$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(E_6 + A_1)}$ .*
- (v) *If  $\Phi_\lambda$  is of type  $A_2 \times E_6$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(2A_2 + A_1)}$ .*
- (vi) *If  $\Phi_\lambda$  is of type  $A_2 \times A_2 \times A_2 \times A_2$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(D_6)}$ .*
- (vii) *If  $\Phi_\lambda$  is of type  $A_4 \times A_4$ , then  $\mathcal{V}_g(\nabla_\zeta(\lambda)) = \overline{\mathcal{O}(A_4 + A_3)}$ .*

PROOF. The cases were verified by using Constrictor Methods I and II. For example, in case (iii), in order to get the inclusion of  $\mathcal{V}_g(\nabla_\zeta(\lambda))$  in  $\overline{\mathcal{O}(E_6(a_3) + A_1)}$ , it suffices to show that  $\mathcal{V}_g(\nabla_\zeta(\lambda)) \cap \mathcal{O}(E_6(a_3)) = \emptyset$  and  $\mathcal{V}_g(\nabla_\zeta(\lambda)) \cap \mathcal{O}(D_6(a_2)) = \emptyset$ . From Theorem 8.7.11, it is enough to prove that  $|w(\Phi_\lambda) \cap \Phi_{J_1}| > 6$  for all  $w \in W$  and  $\Phi_{J_1} \cong E_6$ , and  $|w(\Phi_\lambda) \cap \Phi_{J_2}| > 4$  for all  $w \in W$  and  $\Phi_{J_2} \cong D_6$  which was verified by MAGMA.  $\square$

### 8.13. Support varieties of $\Delta_\zeta(\lambda)$ when $l$ is bad

We remark that given the computation of  $\mathcal{V}_\zeta(\nabla_\zeta(\lambda))$  for  $\lambda \in X_+$  when  $l$  is bad, one can use the same ideas as in the proof of Theorem 8.4.6 to show that when  $l$  is bad then

$$\mathcal{V}_g(\nabla_\zeta(\lambda)) = \mathcal{V}_g(\Delta_\zeta(\lambda))$$

for all  $\lambda \in X_+$ .



## APPENDIX A

### A.1. Tables I

For each odd integer  $l > 1$  which is not equal to a bad prime for  $\Phi$ , the following tables give an element  $w \in W$  and subset  $J \subset \Pi$  such that  $w(\Phi_0) = \Phi_J$ . In fact, the element  $w$  is chosen so that  $w(\Phi_0^+) = \Phi_J^+$ , and identified by means of a reduced expression in terms of the simple reflections  $s_i = s_{\alpha_i}$ . Similar short-hand notation is used to denote the simple roots in  $J$ . Also given is the type of  $\Phi_0$  (or equivalently  $\Phi_J$ ), the Bala-Carter label for the nilpotent orbit  $\mathcal{N}(\Phi_0) = G \cdot \mathfrak{u}_J$ , and the dimension of that orbit.

Type  $F_4$ :

$l$	$\dim \mathcal{N}(\Phi_0)$	$\Phi_0$	$w$	$J$	orbit
5	40	$A_2 \times A_1$	$s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_1 s_2 s_3$	$\{1, 3, 4\}$	$F_4(a_3)$
7	44	$A_1 \times A_1$	$s_2 s_1 s_4 s_3 s_2 s_3 s_4 s_1 s_3 s_2 s_4 s_3 s_2$	$\{1, 3\}$	$F_4(a_2)$
9	46	$A_1$	$s_4 s_2 s_3 s_1 s_2$	$\{3\}$	$F_4(a_1)$
11	46	$A_1$	$s_4 s_2 s_3 s_1 s_2 s_3 s_4$	$\{3\}$	$F_4(a_1)$
$\geq 12$	48	$\emptyset$	—	$\emptyset$	$F_4$

Type  $G_2$ :

$l$	$\dim \mathcal{N}_1(\Phi_0)$	$\Phi_0$	$w$	$J$	orbit
5	10	$A_1$	$s_2 s_1$	$\{1\}$	$G_2(a_1)$
$\geq 6$	12	$\emptyset$	—	$\emptyset$	$G_2$

Type  $E_6$ :

$l$	$\dim \mathcal{N}(\Phi_0)$	$\Phi_0$	$J$	orbit
5	62	$A_2 \times A_1 \times A_1$	$\{1, 2, 3, 5\}$	$A_4 + A_1$
7	66	$A_1 \times A_1 \times A_1$	$\{2, 3, 5\}$	$E_6(a_3)$
9	70	$A_1$	$\{4\}$	$E_6(a_1)$
11	70	$A_1$	$\{4\}$	$E_6(a_1)$
$\geq 12$	72	$\emptyset$	$\emptyset$	$E_6$

$l$	$w$
5	$s_4 s_3 s_2 s_1 s_5 s_4 s_2 s_5 s_6 s_3 s_5 s_6 s_2 s_4 s_1 s_3 s_4 s_3 s_2 s_1 s_4 s_3 s_1 s_4 s_3 s_2 s_1$
7	$s_4 s_2 s_1 s_3 s_4 s_6 s_5 s_4 s_3 s_6 s_1 s_5 s_3 s_2 s_4 s_6 s_5$
9	$s_3 s_1 s_2 s_5 s_4 s_6 s_5 s_3$
11	$s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_3 s_4 s_2$

Type  $E_7$ :

$l$	$\dim \mathcal{N}_1(\Phi_0)$	$\Phi_0$	$J$	orbit
5	106	$A_3 \times A_2 \times A_1$	$\{1, 2, 3, 5, 6, 7\}$	$A_4 + A_2$
7	114	$A_2 \times A_1 \times A_1 \times A_1$	$\{1, 2, 3, 5, 7\}$	$A_6$
9	118	$A_1 \times A_1 \times A_1 \times A_1$	$\{2, 3, 5, 7\}$	$E_6(a_1)$
11	120	$A_1 \times A_1 \times A_1$	$\{2, 3, 5\}$	$E_7(a_3)$
13	122	$A_1 \times A_1$	$\{4, 6\}$	$E_7(a_2)$
15	124	$A_1$	$\{1\}$	$E_7(a_1)$
17	124	$A_1$	$\{1\}$	$E_7(a_1)$
$\geq 18$	126	$\emptyset$	$\emptyset$	$E_7$

$l$	$w$
5	$s_4 s_5 s_3 s_2 s_4 s_2 s_3 s_1 s_3 s_6 s_5 s_4 s_3 s_5 s_6 s_5 s_3 s_2 s_4 s_2 s_3 s_2 s_4 s_6 s_5 s_4 s_7 s_6 s_4 s_3 s_5 s_6$
7	$s_4 s_2 s_3 s_6 s_5 s_4 s_6 s_3 s_1 s_2 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2$
9	$s_4 s_2 s_5 s_4 s_1 s_6 s_5 s_3 s_4 s_5 s_2 s_3 s_4 s_7 s_6 s_1 s_2 s_3 s_4 s_5 s_6 s_7$
11	$s_4 s_3 s_2 s_5 s_6 s_5 s_7 s_6 s_1 s_3 s_4 s_2 s_3 s_5 s_4 s_3 s_1 s_2 s_6 s_5 s_4 s_3$
13	$s_7 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_1 s_2 s_3 s_4 s_5 s_6$
15	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4$
17	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_7 s_6 s_5 s_2 s_4 s_3 s_1$

Type  $E_8$ :

$l$	$\dim \mathcal{N}(\Phi_0)$	$\Phi_0$	$J$	orbit
7	212	$A_4 \times A_2 \times A_1$	$\{1, 2, 3, 5, 6, 7, 8\}$	$A_6 + A_1$
9	220	$A_3 \times A_2 \times A_1$	$\{1, 2, 4, 6, 7, 8\}$	$E_8(b_6)$
11	224	$A_2 \times A_2 \times A_1 \times A_1$	$\{1, 2, 3, 5, 7, 8\}$	$E_8(a_6)$
13	228	$A_2 \times A_1 \times A_1 \times A_1$	$\{2, 3, 5, 6, 8\}$	$E_8(a_5)$
15	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{1, 4, 6, 8\}$	$E_8(a_4)$
17	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{2, 3, 5, 7\}$	$E_8(a_4)$
19	234	$A_1 \times A_1 \times A_1$	$\{2, 3, 5\}$	$E_8(a_3)$
21	236	$A_1 \times A_1$	$\{6, 8\}$	$E_8(a_2)$
23	236	$A_1 \times A_1$	$\{6, 8\}$	$E_8(a_2)$
25	238	$A_1$	$\{1\}$	$E_8(a_1)$
27	238	$A_1$	$\{1\}$	$E_8(a_1)$
29	238	$A_1$	$\{1\}$	$E_8(a_1)$
$\geq 30$	240	$\emptyset$	$\emptyset$	$E_8$

$l$	$w$
7	$s_4 s_3 s_2 s_4 s_5 s_4 s_2 s_1 s_3 s_1 s_4 s_2 s_3 s_5 s_4 s_6 s_5 s_4 s_3 s_1 s_2 s_4 s_3 s_7 s_6 s_8 s_7 s_5$
—	$s_4 s_2 s_6 s_5 s_4 s_3 s_4 s_5 s_6 s_7 s_6 s_8 s_7 s_8 s_1 s_3 s_2 s_4 s_5 s_6 s_4 s_1 s_3 s_2 s_4 s_1 s_2 s_3$
9	$s_5 s_6 s_4 s_5 s_2 s_4 s_3 s_4 s_5 s_6 s_1 s_2 s_3 s_4 s_2 s_5 s_4 s_3 s_4 s_2 s_6 s_5 s_4 s_7 s_6 s_5 s_8 s_7 s_6 s_1 s_3$
—	$s_4 s_2 s_5 s_4 s_3 s_4 s_5 s_6 s_5 s_4 s_1 s_2 s_3 s_4 s_5 s_1 s_3 s_4 s_7 s_6 s_5 s_2 s_4 s_5 s_6 s_7 s_1 s_2 s_3 s_4 s_5$
11	$s_4 s_2 s_5 s_3 s_4 s_3 s_2 s_6 s_5 s_4 s_2 s_7 s_6 s_3 s_5 s_4 s_5 s_6 s_7 s_1 s_3 s_4 s_5 s_2 s_4 s_6 s_5 s_8$
—	$s_7 s_6 s_7 s_1 s_3 s_4 s_5 s_2 s_4 s_2 s_6 s_7 s_3 s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_4 s_3 s_1 s_3 s_8 s_7 s_6 s_5$
13	$s_4 s_2 s_3 s_4 s_5 s_7 s_6 s_8 s_7 s_5 s_4 s_2 s_3 s_4 s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_2 s_6 s_5$
—	$s_4 s_3 s_2 s_5 s_4 s_3 s_1 s_7 s_8 s_6 s_5 s_7 s_6 s_4 s_2 s_5 s_3 s_4 s_5 s_2 s_6 s_3 s_1 s_4 s_3$
15	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_7 s_8 s_6 s_5 s_7 s_6 s_2 s_4 s_5 s_2 s_4 s_6 s_7 s_5 s_6 s_1 s_3 s_8 s_7$
—	$s_4 s_5 s_3 s_4 s_1 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_8 s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4$
17	$s_4 s_2 s_5 s_4 s_6 s_5 s_7 s_6 s_1 s_3 s_4 s_2 s_5 s_4 s_8 s_7 s_6 s_5 s_3 s_4 s_2 s_1 s_3 s_4 s_5 s_6 s_7 s_8 s_4 s_5 s_6 s_7 s_2 s_4 s_5 s_6 s_1 s_3 s_4$
—	$s_2 s_5 s_4 s_3 s_1 s_8 s_7 s_4 s_5 s_3 s_4 s_1 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_8 s_7 s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1$
19	$s_4 s_2 s_3 s_4 s_1 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6 s_7 s_8 s_4 s_5 s_3 s_4$
—	$s_1 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_8 s_7 s_6 s_5 s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7$
21	$s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7 s_1 s_3 s_4 s_5 s_6 s_2 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_6$
—	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7 s_6 s_5$
23	$s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7 s_1 s_3 s_4 s_5 s_6 s_2 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_6 s_7 s_8$
—	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7 s_6 s_5 s_4 s_2$
25	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7 s_6 s_5 s_4 s_2 s_3 s_4$
27	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6$
29	$s_3 s_4 s_2 s_5 s_4 s_3 s_6 s_5 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_1 s_8 s_7 s_6 s_5 s_4 s_2 s_3 s_4 s_5 s_6 s_7 s_8$



## A.2. Tables II

The information listed below for the exceptional Lie algebras is used for justifying the arguments given in Section 4.9. For each  $l$ , the element  $w \in W$  referred to here is the  $w$  identified in the tables of Appendix A.1. All weights (e.g.,  $w \cdot 0$ ) are listed in the weight (or  $\varpi$ ) basis.

For a given  $J$  and  $w$ , the values of  $w \cdot 0$  and  $-w_{0,J}(w \cdot 0)$  are listed. In the arguments of Section 4.9 a weight  $\lambda$  and a sum of coroots  $\delta^\vee$  are introduced. The values of  $\lambda$ ,  $\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle$ , and  $\langle \lambda, \delta^\vee \rangle$  are listed. Finally, for the root systems of type  $E_n$ , a third table lists the value of  $\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$  along with bounds on possible values for an inner product  $\langle x, \alpha^\vee \rangle$  for  $\alpha \in \Pi \setminus J$ . The object  $x$  is introduced in Section 4.9 and represents various possible combinations of sums of positive roots. This information is only needed in the arguments of Section 4.9 for certain values of  $l$ , and only the relevant information is listed.

Type  $F_4$ :

$l$	$J$	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1) J $	$\lambda$	$\langle \lambda, \delta^\vee \rangle$
11	$\{3\}$	$(0, -6, 10, -6)$	$(0, -4, 10, -4)$	10	$(0, -4, 10, -4)$	10
9	$\{3\}$	$(1, -5, 8, -5)$	$(-1, -3, 8, -3)$	8	$(0, -4, 10, -4)$	10
7	$\{1, 3\}$	$(6, -7, 6, -6)$	$(6, -5, 6, 0)$	12	$(4, -5, 9, -3)$	13
5	$\{1, 3, 4\}$	$(4, -8, 4, 4)$	$(4, -4, 4, 4)$	12	$(4, -4, 4, 4)$	12

Type  $G_2$ :

$l$	$J$	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1) J $	$\lambda$	$\langle \lambda, \delta^\vee \rangle$
5	$\{1\}$	$(4, -3)$	$(4, -1)$	4	$(4, -1)$	4

Type  $E_6$ :

$l$	$J$	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$
11	$\{4\}$	$(0, -6, -6, 10, -6, 0)$	$(0, -4, -4, 10, -4, 0)$
9	$\{4\}$	$(1, -5, -5, 8, -5, 1)$	$(-1, -3, -3, 8, -3, -1)$
7	$\{2, 3, 5\}$	$(-3, 6, 6, -11, 6, -6)$	$(-3, 6, 6, -7, 6, 0)$
5	$\{1, 2, 3, 5\}$	$(4, 4, 4, -10, 4, -5)$	$(4, 4, 4, -6, 4, 1)$

$l$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1) J $	$\lambda$	$\langle \lambda, \delta^\vee \rangle$
11	10	$(0, -4, -4, 10, -4, 0)$	10
9	8	$(0, -4, -4, 10, -4, 0)$	10
7	18	$(-2, 6, 6, -7, 6, -2)$	18
5	16	$(3, 6, 2, -6, 6, -2)$	17

$l$	$\alpha$	$\langle x, \alpha^\vee \rangle$	$\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$
7	$\alpha_1$	$-4 \leq * \leq 1$	-3
7	$\alpha_4$	$-7 \leq * \leq -6$	-7
7	$\alpha_6$	$-4 \leq * \leq 1$	0

Type  $E_7$ :

$l$	$J$	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$
17	$\{1\}$	$(16, 0, -11, 0, 0, 0, 0)$	$(16, 0, -5, 0, 0, 0, 0)$
15	$\{1\}$	$(14, 0, -10, 0, 0, 1, 0)$	$(14, 0, -4, 0, 0, 0, 0)$
13	$\{4, 6\}$	$(0, -7, -7, 12, -13, 12, -8)$	$(0, -5, -5, 12, -11, 12, -4)$
11	$\{2, 3, 5\}$	$(-6, 10, 10, -17, 10, -6, 0)$	$(-4, 10, 10, -13, 10, -4, 0)$
9	$\{2, 3, 5, 7\}$	$(-5, 8, 8, -14, 8, -8, 8)$	$(-3, 8, 8, -10, 8, -8, 8)$
7	$\{1, 2, 3, 5, 7\}$	$(6, 6, 6, -14, 6, -7, 6)$	$(6, 6, 6, -10, 6, -5, 6)$
5	$\{1, 2, 3, 5, 6, 7\}$	$(4, 4, 4, -15, 4, 4, 4)$	$(4, 4, 4, -9, 4, 4, 4)$

$l$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1) J $	$\lambda$	$\langle \lambda, \delta^\vee \rangle$
17	16	$(16, 0, -5, 0, 0, 0, 0)$	16
15	14	$(16, 0, -5, 0, 0, 0, 0)$	16
13	24	$(0, -5, -5, 12, -11, 12, -4)$	24
11	30	$(-4, 10, 10, -13, 10, -4)$	30
9	32	$(-4, 8, 10, -11, 8, -7, 8)$	34
7	30	$(5, 8, 4, -10, 8, -7, 8)$	33
5	24	$(5, 10, 4, -10, 3, 0, 5)$	27

$l$	$\alpha$	$\langle x, \alpha^\vee \rangle$	$\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$
15	$\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7$	$-10 \leq * \leq 12$	0
15	$\alpha_3$	$-15 \leq * \leq 4$	-4
9	$\alpha_1$	$-8 \leq * \leq 4$	-3
9	$\alpha_4$	$-13 \leq * \leq -6$	-10
9	$\alpha_6$	$-12 \leq * \leq 0$	-8

Type  $E_8$ :

$l$	$J$	$w \cdot 0$	$-w_{0,J}(w \cdot 0)$
29	$\{1\}$	$(28, 0, -17, 0, 0, 0, 0, 0)$	$(28, 0, -11, 0, 0, 0, 0, 0)$
27	$\{1\}$	$(26, 0, -17, 1, 0, 0, 0, 0)$	$(26, 0, -9, -1, 0, 0, 0, 0)$
25	$\{1\}$	$(24, 0, -15, 0, 0, 1, 0, 0)$	$(24, 0, -9, 0, 0, -1, 0, 0)$
23	$\{6, 8\}$	$(0, 0, 0, 0, -13, 22, -25, 22)$	$(0, 0, 0, 0, -9, 22, -19, 22)$
21	$\{6, 8\}$	$(2, 0, 0, 0, -13, 20, -21, 20)$	$(-2, 0, 0, 0, -7, 20, -19, 20)$
19	$\{2, 3, 5\}$	$(-10, 18, 18, -29, 18, -13, 4, 0)$	$(-8, 18, 18, -25, 18, -5, -4, 0)$
17	$\{2, 3, 5, 7\}$	$(-14, 16, 16, -25, 16, -16, 16, -11)$	$(-2, 16, 16, -23, 16, -16, 16, -5)$
15	$\{1, 4, 6, 8\}$	$(14, -10, -17, 14, -13, 14, -13, 14)$	$(14, -4, -11, 14, -15, 14, -15, 14)$
13	$\{2, 3, 5, 6, 8\}$	$(-7, 12, 12, -26, 12, 12, -21, 12)$	$(-5, 12, 12, -22, 12, 12, -15, 12)$
11	$\{1, 2, 3, 5, 7, 8\}$	$(10, 10, 10, -22, 10, -18, 10, 10)$	$(10, 10, 10, -18, 10, -12, 10, 10)$
9	$\{1, 2, 4, 6, 7, 8\}$	$(8, 8, -13, 8, -24, 8, 8, 8)$	$(8, 8, -11, 8, -16, 8, 8, 8)$
7	$\{1, 2, 3, 5, 6, 7, 8\}$	$(6, 6, 6, -25, 6, 6, 6, 6)$	$(6, 6, 6, -17, 6, 6, 6, 6)$

$l$	$\langle -w_{0,J}(w \cdot 0), \delta^\vee \rangle = (l-1) J $	$\lambda$	$\langle \lambda, \delta^\vee \rangle$
29	28	$(28, 0, -11, 0, 0, 0, 0, 0)$	28
27	26	$(28, 0, -11, 0, 0, 0, 0, 0)$	28
25	24	$(28, 0, -11, 0, 0, 0, 0, 0)$	28
23	44	$(0, 0, 0, 0, -8, 22, -21, 22)$	44
21	40	$(0, 0, 0, 0, -8, 22, -21, 22)$	44
19	54	$(-8, 18, 18, -25, 18, -8, 0, 0)$	54
17	64	$(-7, 16, 16, -22, 16, -15, 16, -6)$	64
15	56	$(16, -5, -15, 16, -15, 16, -15, 16)$	64
13	60	$(-7, 16, 16, -21, 7, 8, -14, 16)$	63
11	60	$(8, 16, 7, -21, 16, -14, 8, 7)$	62
9	48	$(18, 7, -16, 10, -14, 6, 0, 10)$	51
7	42	$(9, 18, 8, -21, 6, 0, 0, 9)$	50

$l$	$\alpha$	$\langle x, \alpha^\vee \rangle$	$\langle -w_{0,J}(w \cdot 0), \alpha^\vee \rangle$
27	$\alpha_2, \alpha_5, \alpha_6, \alpha_7, \alpha_8$	$-18 \leq * \leq 20$	0
27	$\alpha_3$	$-27 \leq * \leq 4$	-9
27	$\alpha_4$	$-18 \leq * \leq 20$	-1
25	$\alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_8$	$-20 \leq * \leq 22$	0
25	$\alpha_3$	$-27 \leq * \leq 6$	-9
25	$\alpha_6$	$-20 \leq * \leq 22$	-1
23	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	$-10 \leq * \leq 12$	0
23	$\alpha_5$	$-20 \leq * \leq 1$	-9
23	$\alpha_7$	$-26 \leq * \leq -14$	-19
21	$\alpha_1$	$-14 \leq * \leq 16$	-2
21	$\alpha_2, \alpha_3, \alpha_4$	$-14 \leq * \leq 16$	0
21	$\alpha_5$	$-24 \leq * \leq 6$	-7
21	$\alpha_7$	$-26 \leq * \leq -9$	-19
19	$\alpha_1$	$-16 \leq * \leq 1$	-8
19	$\alpha_4$	$-25 \leq * \leq -24$	-25
19	$\alpha_6$	$-16 \leq * \leq 1$	-5
19	$\alpha_7$	$-10 \leq * \leq 12$	-4
19	$\alpha_8$	$-10 \leq * \leq 12$	0
17	$\alpha_1$	$-14 \leq * \leq 1$	-2
17	$\alpha_4$	$-25 \leq * \leq -18$	-23
17	$\alpha_6$	$-20 \leq * \leq -8$	-16
17	$\alpha_8$	$-14 \leq * \leq 1$	-5
13	$\alpha_1$	$-17 \leq * \leq 5$	-5
13	$\alpha_4$	$-24 \leq * \leq -13$	-22
13	$\alpha_7$	$-22 \leq * \leq -3$	-15
11	$\alpha_4$	$-24 \leq * \leq -15$	-18
11	$\alpha_6$	$-21 \leq * \leq -5$	-12



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